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# On the Domination Chain of $m$ by $n$ Chess Graphs

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# On the Domination Chain of $m \times n$ Chess Graphs

A Thesis

Presented to

the Faculty of the Department of Mathematics and Statistics

Murray State University

Murray, Kentucky

In Partial Fulfillment

of the Requirements for the Degree

of Master of Science

by

Kathleen G. Johnson

May 2, 2018

# On the Domination Chain of $m \times n$ Chess Graphs

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## Abstract

The game of chess has fascinated people for hundreds of years in many countries across the globe. Chess is one of the most challenging and well-studied games of skill. The underlying aspects of chess give rise to many classically studied puzzles on the chessboard. Graph theoretic analysis has been used to study numerous chess-related questions. We relay results on various non-attacking packings and coverings for the rook, bishop, king, queen, and knight on square and nonsquare chessboards. These results lay the foundation for our work with the  $m \times n$  Bishop graph. We examine the role of the bishop in the non-attacking packing problem as well as the covering problem for oblong chessboards and present constructions for the domination number and independence number of such graphs.

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# Chapter 1

## Introduction

The strongest chess players were often considered worldwide champions as far back as the Middle Ages. Since chess is a game of strategy, mathematicians have endeavored to build algorithms that would allow anyone or any machine to play a successful game against such a champion since at least the 18<sup>th</sup> century. One famous outcome of this ambition is the Turk as told by Schaffer [17]. This automaton built in 1770 could successfully play chess against a human player and complete the mathematical chess puzzle known as the Knight's Tour. The Turk itself went on exhibition across Europe and North America playing against strong chess players for almost 84 years. The Turk was confirmed to be a fake – that is, a person hiding inside the machine playing the game, by the son of the last owner Mitchell [15], in 1857. The fact that this was not revealed for almost one hundred years speaks to our desire to find a formulation that can do anything we want it to do. The computer chess champion ambition came to fruition in 1997 when chess-playing computer Deep Blue beat a current world chess champion.

While many problems involving chess have been solved the game still has numerous puzzles for which we have yet to produce a solution. The algorithmic nature of chess lends itself naturally to mathematically-inclined puzzles. In particular, the movement



of the pieces have inspired numerous problems, many of which are still unsolved today. Even as older problems are solved, new ones take their place as we imagine chess on different types of chessboards and enact new methods of play. In our efforts to solve these problems in the present, we have mathematical tools such as graph theory to describe and discover solutions to existing problems.

Our purpose is to examine and provide new information involving classic graph theoretic chess problems. We examine the covering problem, non-attacking covering problem, and packing problem, among others. These problems are applied to various chess pieces on various shaped chessboards. The chess pieces studied are five of the six classic pieces: the queen, the king, rooks, bishops, and knights. While the above mentioned problems have been studied at length for  $n \times n$  boards not much is known for  $m \times n$  boards. We will take this avenue in our research, focusing on the packing and covering problems for the bishop.

In Chapter 2 we lay the groundwork for our use of chess, including the layout of the board and the movement of the five pieces under consideration. In Chapter 3 we begin our discussion of graph theory, drawing together the necessary definitions with our perspective of chess graph and the domination chain. In Chapter 4 we examine known results for the domination chain for the rook, bishop, king, queen, and knight graphs. Chapter 5 contains our findings from examining the movement of the bishop on a rectangular board, right triangular board, and trapezoidal board.

# Chapter 2

## Chess Games

### 2.1 A Brief History of Chess

Chess is an ancient game that began in 6<sup>th</sup> century India and spread throughout the world as it evolved into the form we know today [16]. Chess is a war game played between two players. The two players begin the game with the same pieces and settings so that the players must use tactics and strategy to win.

Modern chess developed during the Middle Ages. As chess became more popular in Europe, so did chess-based puzzles. Watkins [18] writes that the earliest chessboard puzzle he knows of is Guarini's Problem from 1512. Several famous mathematicians have worked on various chess puzzles, including Leonhard Euler and Carl Friedrich Gauss who worked on The Knight's Tour and Eight Queens Problem, respectively.

Queens Domination is another popular problem. Problems about the queen piece are well-studied and have been expanded into the other chess pieces. Problems on an  $n \times n$  board have been thoroughly considered. We examine some less well investigated problems on the  $m \times n$  board.

Mathematicians and computer scientists have used computer algorithms to resolve chess problems since the mid-twentieth century.

## 2.2 Chessboard Layout

A standard chessboard is the  $8 \times 8$  checkerboard. This board consists of eight columns and eight rows made up of an equal number of alternating black and white squares. There are 32 chess pieces to start a standard game of chess, 16 white pieces and 16 black pieces, covering half of the standard 64 squares. Of each color set, there is one queen ♔, one king ♚, two bishops ♗, two knights ♞, two rooks ♖, and eight pawns ♟. At the beginning of a standard game, each player lays out his/her pieces as shown in Figure 2.2.1.

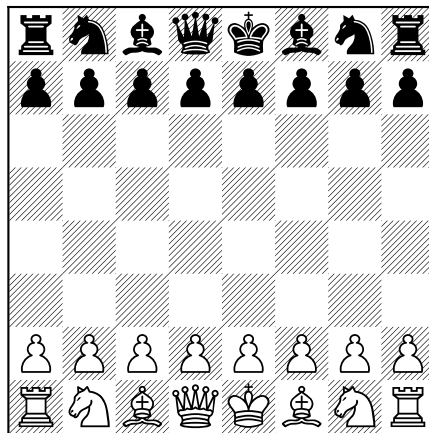
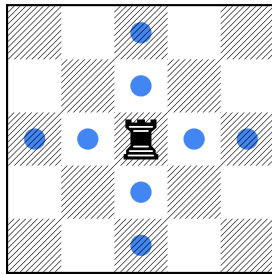


Figure 2.2.1: The layout of a standard game of chess. Note that we will follow the convention of a black square in the lower left corner of the board.

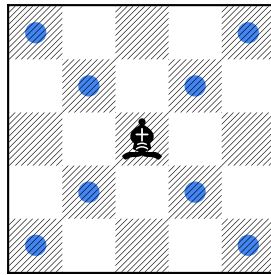
## 2.3 Chess-piece Moves

While the classic game of chess raises some interesting mathematical questions, our focus will be more directed toward the movement of the pieces. Each piece has a distinct set of moves that it can make. A rook can move any number of squares either horizontally or vertically, while a bishop can move any distance diagonally. A knight can move on a chessboard by going two squares in any horizontal or vertical

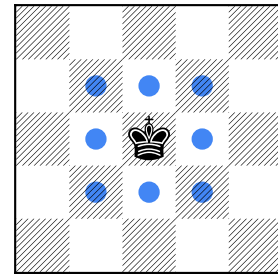
direction, and then turning either left or right one more square [18], thus moving in an “L”. The king’s move is more restrictive: it may move only one square horizontally, vertically, or diagonally. The queen has the most control over the board as it can move any number of squares horizontally, vertically, or diagonally, thus making it a combination of bishop and rook movement. We will not discuss the movement options for the pawn as it is dependent on the current layout of the board. The above movement of these pieces can be seen in Figure 2.3.1. Notice in Figure 2.3.1b that a bishop has a restriction on its location: it can only ever move to squares of its original starting color. That is, a bishop on white can only control white squares and a bishop on black can only control black squares.



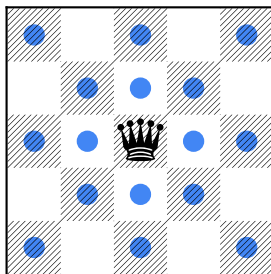
(a) Rook



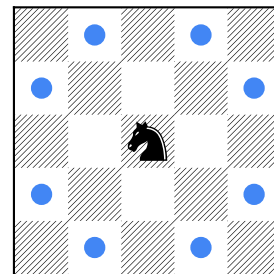
(b) Bishop



(c) King



(d) Queen



(e) Knight

Figure 2.3.1: Five chess pieces and their moves.

From Figures 2.3.1c and 2.3.1e we can see both the king and the knight has the

possibility to attack, or control, at most eight squares, while the limiting factor for the attacking options of a rook, bishop, or queen (Figures 2.3.1a, 2.3.1b, and 2.3.1d) are determined by the size of the board. Notice that the queen can attack more squares than the rook or bishop alone, since its movement is a combination of both the bishop and the rook motions.

## 2.4 Chess Variants

As chess evolved in different countries, it took a variety of different forms. It began as the Indian game Chaturanga and now several different versions of the game exist in Asia. A particularly notable chess-variant is Shogi from Japan.

Shogi is played on a  $9 \times 9$  board with five of the classic chess pieces (rook, bishop, king, knight, pawn) as well as some additional pieces (gold and silver generals, and lance). A major difference is that certain pieces can be promoted and acquire additional movement. The rook piece advances to become the dragon king and the bishop piece can become the dragon horse. These advanced motions make these pieces the most similar to the classic queen piece than any of the other five standard pieces. We will see later in Chapter 4 these pieces have been studied as part of further research into Queens Domination.

## 2.5 Variations on Modern Chess

Modern chess can be played on different surfaces as well. Besides a rectangular board, including those that are square, chess movements have been analyzed on a torus, various 3D-boards, triangular boards [18], boards with hexagons in the place of squares [6], just to name a few. Each of these board shapes has given rise to different results of the more classically studied chess problems.

# Chapter 3

## Graph Theory Definitions

### 3.1 Basic Terminology

Various graph theory parameters have been applied to the chess graphs constructed by the movements of each of the mentioned five chess pieces. To gain insight into these constructions, we begin with the necessary graph theory definitions.

**Definition 3.1.1.** A *graph* is a pair  $G = (V, E)$  consisting of a *vertex set*  $V(G)$  (or  $V$  when the graph is understood) together with an *edge set*  $E(G)$  (or  $E$ ), which is comprised of 2-element subsets of  $V$ , the *endpoints* of an edge.

Two vertices that form an edge are *adjacent* vertices and so they are *neighbors*. The *order* of a graph  $G$  denoted  $n(G)$  refers to the number of vertices and the *size* of the graph  $e(G)$  is the number of edges. The *degree* of a vertex  $v$  is the number of edges with  $v$  as at least one of its endpoints. See Figure 3.1.1 for an example of a graph and these parameters.

When working with a graph we often need to focus on a certain subset of the graph. Such a subset is more formally set out in the following definitions:

**Definition 3.1.2.** A *subgraph* has vertices and edges belonging to  $G$ .

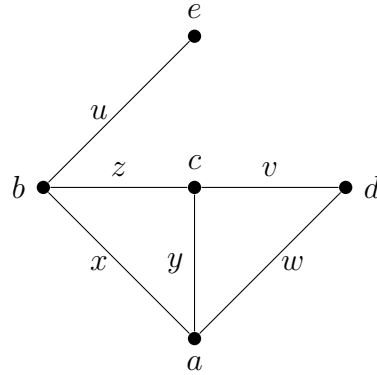
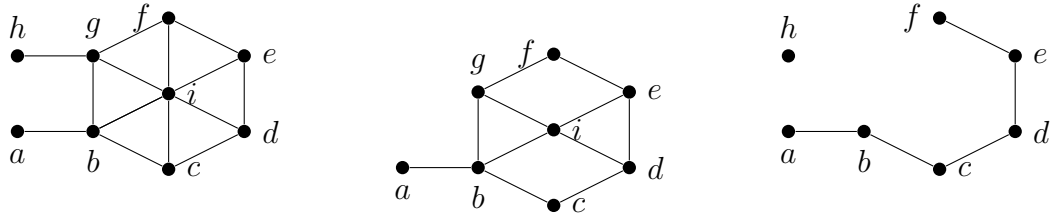


Figure 3.1.1: Graph  $G$  with vertex set  $V(G) = \{a, b, c, d, e\}$  and edge set  $E(G) = \{u, v, w, x, y, z\}$ .  $n(G) = 5$  and  $e(G) = 6$  with the vertices  $a, b,$  and  $c$  having the largest degree of 3.  $a$  and  $b$  are adjacent and are said to be neighbors as they are the endpoints of edge  $x$ .

**Definition 3.1.3.** An *induced subgraph*  $G[A]$  has vertex set  $A \subseteq V(G)$  obtained by taking  $A$  and all edges of  $G$  having both endpoints in  $A$ .

Figure 3.1.2 gives an example of a graph with a subgraph and induced subgraph.



(a) A graph  $G$  with vertex set  $V = \{a, b, c, d, e, f, g, h, i\}$ . (b) A subgraph  $H$  of  $G$ . (c) An induced subgraph  $G[A]$  with vertex set  $A = \{a, b, c, d, e, f, h\}$ .

Figure 3.1.2: A graph  $G$  with induced subgraph  $H$  and induced subgraph  $G[A]$ .

**Definition 3.1.4.** A *component* is a maximal connected subgraph.

**Definition 3.1.5.** A *connected* graph  $G$  has a  $u, v$ -path for each set of distinct vertices  $u, v$ . That is every vertex  $u$ , has a path to every vertex  $v, u, v \in V(G), u \neq v$ .

**Definition 3.1.6.** A *bipartite* graph  $G$  has vertex sets  $X$  and  $Y$  such that each edge in  $G$  has one vertex in  $X$  and one vertex in  $Y$ .

Observe an example a bipartite graph that is connected in Figure 3.1.3a and of a disconnected graph in Figure 3.1.3b The connected graph is comprised of one component while the disconnected graph is composed of three components, one of which is a singleton vertex.

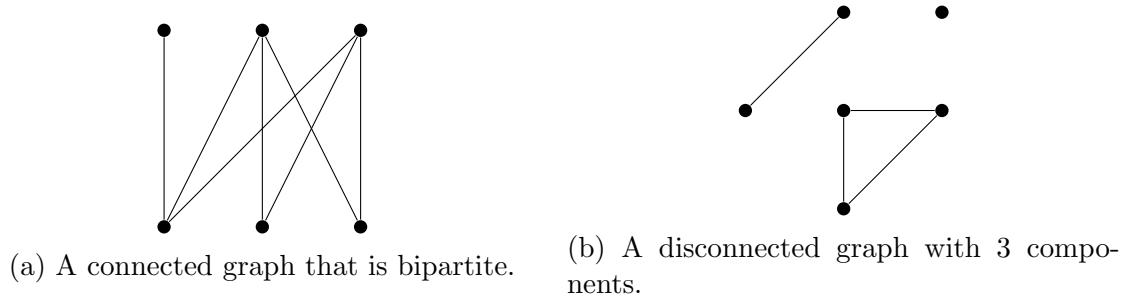


Figure 3.1.3: An example of a bipartite graph that is connected and a disconnected graph.

Of particular importance in this research is the claw graph. The claw graph, more formally known as  $K_{1,3}$ , consists of one vertex, often depicted in the middle of the graph, which is adjacent to the remaining three vertices in the graph. This graph is depicted in Figure 3.1.4.

**Definition 3.1.7.** A graph is considered *claw-free* if it does not contain the claw graph as an induced subgraph.

Note that the graph  $H$  given in Figure 3.1.2b is not claw-free as the induced subgraph  $H[S]$  with  $S = \{a, b, g, c\}$  is a claw, while the graph  $G[A]$  in Figure 3.1.2c is, in fact, claw-free.



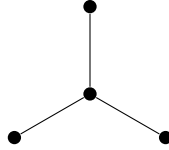


Figure 3.1.4: The claw graph.

## 3.2 Packing and Covering Parameters

Our research is focused on various packing and covering parameters of a graph. Specifically, these covering and packing problems will often be focused on the domination or independence of vertices, causing us to minimize or maximize our vertex selection, respectively. As a note the “or” between domination and independence is not exclusive – we will also be considering vertex independent domination which must satisfy the requirements of both parameters. From there, we will consider one other related parameter, the irredundance number, and connect these ideas with the domination chain.

Consider the following question: determine a set of vertices so that every vertex is adjacent to at least one vertex in this set. This idea of a covering problem lends itself to be viewed as a domination parameter. Domination problems are well-studied in graph theory due to their practical applications. For example, Watkins [18] draws a connection between chess and domination in that chess began as a game of war. Thus it is of no surprise that the idea of domination and chess go together.

**Definition 3.2.1.** A *dominating set* is a set  $S \subseteq V$  such that every vertex outside  $S$  has a neighbor in  $S$ . That is, every vertex not in  $S$  is adjacent to a vertex in  $S$ .

Clearly we can make this set as large as we wish, up to including all of the vertices in our graph. Thus, the challenge becomes determining how small we can make  $S$ : a minimal dominating set that has the fewest vertices needed to dominate the graph.

**Definition 3.2.2.** The *domination number* of a graph  $\gamma(G)$  is the minimum cardinality of a minimal dominating set of vertices.

A dominating set of size  $\gamma(G)$  is known as a *minimum dominating set*.

**Definition 3.2.3.** The *upper domination number* of a graph  $\Gamma(G)$  is the maximum cardinality of a minimal dominating set.

A dominating set of size  $\Gamma(G)$  is known as a *maximum dominating set*.

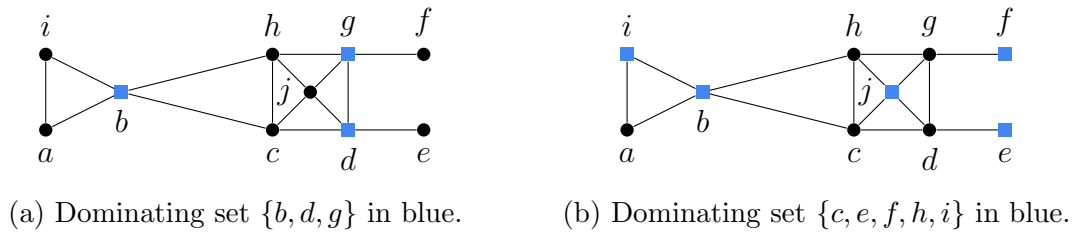


Figure 3.2.1: Dominating sets for a graph  $L$ .

In Figure 3.2.1a we see that a minimum dominating set is  $\{b, d, g\}$ , making the domination number  $\gamma(L) = 3$ . There is a maximum dominating set  $\{b, e, f, i, j\}$ , shown in blue in Figure 3.2.1b, giving the upper domination number  $\Gamma(L) = 5$ .

Given a graph, how many vertices can we choose so that none of them are neighbors? Let us formally define this idea.

**Definition 3.2.4.** ([20]) An *independent set* is a set of pairwise nonadjacent vertices.

For this parameter it is easy to see that we can make our set of vertices as small as we wish – we could select the empty set of vertices and still satisfy the above definition. Therefore, we will seek out the largest possible set of vertices that still satisfies the given condition: a maximal independent set has a set of vertices such that the addition of another vertex would cause the set to be not independent.

**Definition 3.2.5.** ([20]) The *independence number* of a graph  $\alpha(G)$  is the maximum size of an independent set of vertices.



(a) A graph  $M$  with independent set  $\{b, d, e\}$  in blue. (b) A graph  $N$  with independent dominating set  $\{c, f\}$  in blue.

Figure 3.2.2: An example of independent and independent dominating sets.

We may require a set of vertices to be both dominant and independent. This idea leads to the next parameter.

**Definition 3.2.6.** ([13]) The *independent domination number*  $i(G)$  of a graph is the minimum cardinality of an independent dominating set.

Gross, et al. [10] defined the class of domination perfect graphs through its induced subgraphs, while Goddard found the same connection through the exclusion of the claw graph.

**Definition 3.2.7.** [10] A graph  $G$  is *domination perfect* if for every induced subgraph  $H$ ,  $\gamma(H) = i(H)$ .

**Theorem 3.2.8.** [9] If a graph  $G$  is claw-free graph, then  $\gamma(G) = i(G)$ .

**Corollary 3.2.9.** [9] If a graph  $G$  is claw-free graph, then  $G$  is domination perfect.

Several families of graphs have their own properties. For instance the *cycle graph* family  $C_n$ , in which  $C_6$  is shown in Figure (3.2.2a), has independence number  $\alpha(C_n) = \lfloor \frac{n}{2} \rfloor$  where  $n$  denotes the order of  $C_n$ . In Figure (3.2.1a), we see that our dominating set is not independent, and so our independent domination number  $i(L)$  must be greater than or equal to our domination number  $\gamma(L)$ .

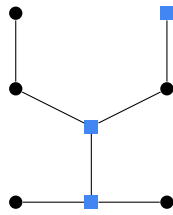
**Definition 3.2.10.** ([13]) A set  $S$  of vertices in a graph is called an *irredundant set* if for each vertex  $v \in S$  either  $v$  itself is not adjacent to any other vertex in  $S$  or else

there is at least one vertex  $u \notin S$  such that  $u$  is adjacent to  $v$  but to no other vertex in  $S$ . That is,  $v$  has a *private neighbor* or is a private neighbor itself.

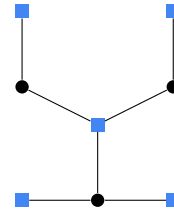
The difficulty in creating an irredundant set comes when making the set as large as possible. Optimizing this packing problem can be done in two different ways, as set out in the following definitions.

**Definition 3.2.11.** ([13]) The *irredundance number* of a graph  $ir(G)$  is the minimum cardinality of a maximal irredundant set of vertices.

**Definition 3.2.12.** ([13]) The *upper irredundance number* of a graph  $IR(G)$  is the maximum cardinality of an irredundant set of vertices.



(a) An irredundant set of size 3. Note that this set is not dominating.



(b) An irredundant set of size 5.

Figure 3.2.3: Irredundance graphs for  $W$ . Note that  $ir(W) = 3$  as seen on the left while  $IR(W) = 5$ , shown on the right.

As stated in Chapter 1, we will examine the covering problem, non-attacking covering problem, and packing problem for various chess pieces. In the context of graph theory we now refer to them as domination, independent domination, and independence problems. We study these parameters as well as the other covering and packing parameters: irredundance, upper domination and upper irredundance. The following holds true for any graph  $G$  and is known as a domination chain according to Gross, Yellen, and Zhang [10].

**Theorem 3.2.13.** ([10])  $ir(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq IR(G)$

The first three values in the chain minimize sets and the last three maximize sets. For certain classifications of graphs, these three minimization parameters,  $ir(G)$ ,  $\gamma(G)$ , and  $i(G)$  tend to be equal and the same can be said for the maximized values  $\alpha(G)$ ,  $\Gamma(G)$ , and  $IR(G)$ . In fact, for a bipartite graph, Cockayne et al. [4] proved that the maximized values are the same.

**Theorem 3.2.14.** ([4]) *If  $G$  is a bipartite graph, then  $\alpha(G) = \Gamma(G) = IR(G)$ .*

Watkins [18] takes an intuitive approach to the relationship between the domination chain parameters. For the minimizing numbers, as we add additional constraints to finding a minimum set of vertices, such a set can only increase in size.  $ir(G) \leq \gamma(G)$  since a dominating set is an irredundant set that also dominates, and, hence, the extra conditions can only make a required set larger. Transitioning to consider the second and third parameters, we find that adding the requirement of independence yields  $\gamma(G) \leq i(G)$  similarly. For our maximization concerning the remaining three parameters, we start with the restriction that our vertices must be independent, limiting the maximum size of our set. Therefore,  $\alpha(G) \leq \Gamma(G)$  as the independence restriction is removed. Similarly,  $\Gamma(G) \leq IR(G)$  since the dominating restriction is removed when considering the upper irredundance number.

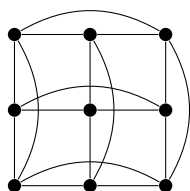
Domination and independence numbers are the most popular of the six to study as they have wide-ranging practical applications. Among these six parameters the irredundance numbers are the most difficult to study and, subsequently, have the least number of known results.

### 3.3 Chess Graphs

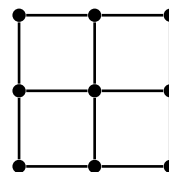
We will now combine the ideas of chess piece movements and graph theory. As noted in [18], the chessboard and motion of the pieces are represented very well using graphs. Each chess piece makes its own graph using a vertex to represent a single square on

the board and edges to represent the movement of the piece from a certain square to other allowable squares. In this way we can make graphs for each piece that look similar to the figures in Section 2.3. Notice that for each chess piece, their graphs will all have the same order when we fix the board size. The difference between the figures given before and these graphs is that the graph shows every possible move from every possible square on the board. This will cause the graph to have a large amount of edges, which often makes its visualization difficult. To simplify the graph, we will use the convention of varied line thicknesses: when all the edges are drawn, thinner edges will be used; when thicker edges are drawn vertices are adjacent if they lie on the same line (vertical, horizontal or diagonal). We begin with the rook graph.

The Rook graph is the most straightforward graph, consisting of horizontal and vertical edges. This is due to the motion of the rook, which allows the piece to move any number of spaces along its current row or column on the board.



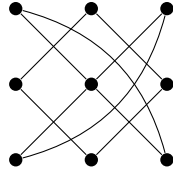
(a) Rook graph



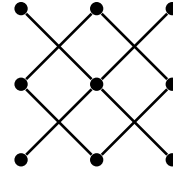
(b) Simplified Rook Graph

Figure 3.3.1: The  $3 \times 3$  Rook board represented as a graph and a simplified graph.

A  $3 \times 3$  Bishop graph has only edges representing diagonal motion as seen earlier in Section 2.3. A Bishop graph on more than one vertex is always disconnected and, specifically, a Bishop graph on a single row or column is comprised only of isolated vertices. For graphs on more than one row or column, the black and white squares each form their own components of the graph since a bishop moving diagonally can only travel across same-colored squares. Because of this separability, we will often only consider one component at a time in our analysis.



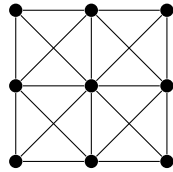
(a) Bishop graph



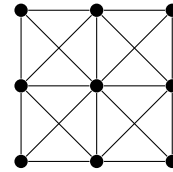
(b) Simplified Bishop Graph

Figure 3.3.2: The  $3 \times 3$  Bishop board represented as a graph and a simplified graph.

Recall that the movement of the king allows the piece to move only one square in any of eight possible directions. Since the piece is restricted to moving only one square at a time, its simplified graph is identical to the original graph.



(a) King graph



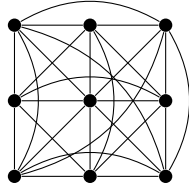
(b) Simplified King Graph

Figure 3.3.3: This is the  $3 \times 3$  King board represented as a graph and then “simplified”. Note that since the king can move only one space in any direction no simplified versions of edges can be used.

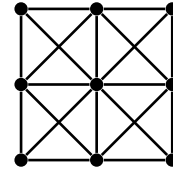
Among all standard chess pieces, the queen has the most freedom in its movement. Because of this, its graph will be the most complicated. The queen combines the motions of a rook and a bishop. Also, the Rook, Bishop, and King graphs are all subgraphs of the Queen graph for a given board size.

We can see in Figures 3.3.1, 3.3.2, and 3.3.4 that the pieces that can move more than one square in any direction have more complicated graphs and this complication will grow faster as the size of the board increases. Hence, it is easier to analyze the simplified graphs, remembering that the rook, bishop, and queen can move any distance along a straight path.

The Knight graph is already simplified as it cannot move more than one “L”



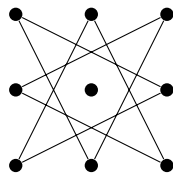
(a) Queen graph



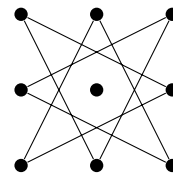
(b) Simplified Queen Graph

Figure 3.3.4: The  $3 \times 3$  Queen board represented as a graph and a simplified graph.

shape at a time. The Knight graph can be disconnected depending on the number of squares in the original board. In fact, it is known to always be disconnected if the board contains either one or two rows or columns. The  $3 \times 3$  example given in Figure 3.3.5 has an isolated vertex in the center, since a knight starting at this position cannot travel two squares out in any one direction and, thus, cannot complete an “L” motion. The Knight graph is also bipartite since in its “L” motion, it moves two squares over, to the same colored square and then over one more either left or right, both of which must be opposite colored squares. Thus, the Knight graph has two partitions: the black squares and the white squares.



(a) Knight graph



(b) Simplified Knight Graph

Figure 3.3.5: This is the  $3 \times 3$  Knight board represented as a graph and then “simplified”.



# Chapter 4

## Known Results

In the following sections we will discuss some of the known domination chain results for each of the five major chess pieces. This short survey encompasses known results from chess piece graphs on both  $n \times n$  and  $m \times n$  boards. Some of the earliest work was done by Yaglom and Yaglom [2] for  $n \times n$  King, Bishop, and Rook graphs. Two previous  $n \times n$  surveys by Fricke et al. [8] and Haynes, Hedetniemi, and Slater [12] have filled in the gaps in the Bishop and Rook graphs such that the whole of the six-parameter domination chain is known for each of these pieces. These surveys also provide additional results for the other chess pieces. When parameters are not known we give bounds, especially in the case of the Queen graph.

Problems solved by Yaglom and Yaglom [2] are the proofs of the theorems stated in Fricke et al. [8] and Haynes, Hedetniemi, and Slater [12].

### 4.1 Rook

As noted above, every parameter in our domination chain is known for  $R(n, n) = R_n$ . From the motion of the rook piece it is easy to see that  $n$  rooks are needed to cover a  $n \times n$  Rook graph as seen in Figure 4.1.1a.

**Theorem 4.1.1.** ([8]) For  $n \geq 31$ ,  $ir(R_n) = n$ .

**Theorem 4.1.2.** ([2], [8]) For  $n \geq 1$ ,  $\gamma(R_n) = i(R_n) = \alpha(R_n) = n$ .

One simple construction for finding a dominating set is to select a set of vertices corresponding to a single row or column on the given square chessboard (Figure 4.1.1a); to find an independent dominating set, we, for example, may instead choose those vertices on any one of the two main diagonals (Figure 4.1.1b).

$R_n$  extends naturally into  $R(m, n)$  for some results as the Rook has unlimited horizontal and vertical movement, so extending the board by one row or column still ensures it is covered. For example, we can still find a dominating set by selecting either an entire row or column from the board. However, to minimize this parameter, as is needed to determine  $\gamma(R(m, n))$ , we would select the smaller of the two possibilities: if there are more columns than rows, select a set of vertices corresponding to a single column and vice versa if the number of rows is greater than (or equal to) the number of columns.

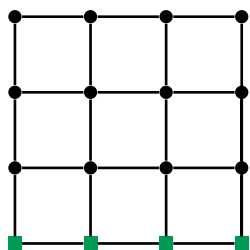
Similar to Theorem 4.1.2, equality among the three minimizing parameters in the domination chain are all equal for  $R(m, n)$ . See Figure 4.1.2 for an example on a  $4 \times 3$  graph.

**Corollary 4.1.3.** [14] For  $n \geq 1$ ,  $\gamma(R(m, n)) = i(R(m, n)) = \alpha(R(m, n)) = \min\{m, n\}$ .

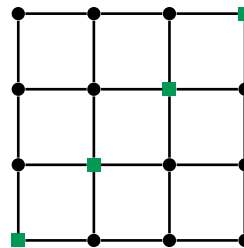
The straightforward nature of the Rook graph yields that every parameter for  $R_n$  has value  $n$  except the upper irredundance number  $IR(R_n)$ . We see that  $IR(R_n) = n$  up to  $n = 4$ , but quickly grows past this value as  $n$  gets large. An example of the general construction for  $IR(R_n)$  for  $n \geq 4$  is shown in Figure 4.1.3.

**Theorem 4.1.4.** ([8]) For  $n \geq 1$ ,  $\Gamma(R_n) = n$ .

**Theorem 4.1.5.** ([8]) For  $n \geq 4$ ,  $IR(R_n) = 2n - 4$ .



(a)  $\gamma(R_4) = 4$  with dominating set in green along the bottom row.



(b)  $\alpha(R_4)$  with independent set in green along a main diagonal. This set is also an independent dominating set.

Figure 4.1.1: The domination number and independence number of an  $n \times n$  Rook graph are equal.

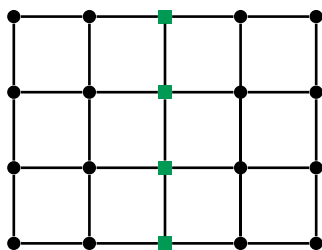


Figure 4.1.2: The domination number of  $R(4, 5)$  is 4.

## 4.2 Bishop

Every parameter in our domination chain is known for  $B(n, n) = B_n$ , as seen in Fricke et al.'s survey [8]. Bishop graphs are popularly analyzed as Rook graphs rotated 45 degrees because Rook graphs are simpler to examine as noted in Section 4.1.

As seen in Figures 4.2.1c and 4.2.1d we can take the smallest whole  $n \times n$  Rook graph that is embedded for each rotated graph to find the lower bound for the domination number of a Bishop graph.

Using results from Cockayne, Gamble, and Shepherd [5] and Yaglom and Yaglom [2], we have that the lower three parameters in the domination chain, irredundance number, domination number, and independent domination number, are all equal.

**Theorem 4.2.1.** ([8]) For  $n \geq 31$ ,  $ir(B_n) = n$ .

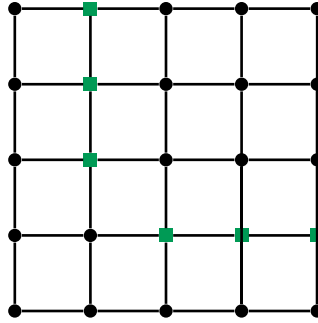


Figure 4.1.3: The upper irredundance number of  $B_5$  is  $2(5) - 4 = 6$  with irredundant set in green.

**Theorem 4.2.2.** ([2], [5]) For  $n \geq 1$ ,  $\gamma(B_n) = n$ .

**Corollary 4.2.3.** ([2], [5]) For  $n \geq 1$ ,  $i(B_n) = n$ .

Yaglom and Yaglom showed that the chosen vertices that comprise an maximum independent set are forced to be picked from those with smaller degree. This can be seen in Figure 4.2.2.

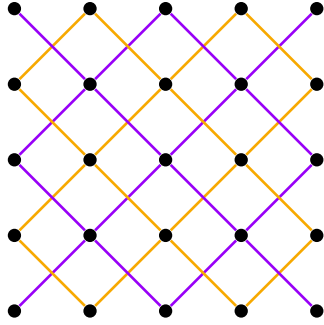
**Theorem 4.2.4.** ([2], [18]) All of the bishops in an independent set of maximum size on  $B_n$  are on the outer ring of squares.

While the the independence number and upper domination number are equal for the Bishop graph on an  $n \times n$  board, the upper irredundance number can be shown to be much larger and follows an interesting pattern.

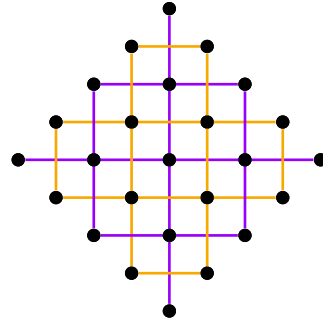
**Theorem 4.2.5.** ([2], [8]) For  $n \geq 1$ ,  $\alpha(B_n) = \Gamma(B_n) = 2n - 2$ .

The following result is due to Fricke et al. [8], though a minor correction is needed. In order to satisfy that  $\Gamma(G) \leq IR(G)$  in the domination chain, we must have  $n \geq 6$ . An example on a  $6 \times 6$  board can be seen in Figure 4.2.4.

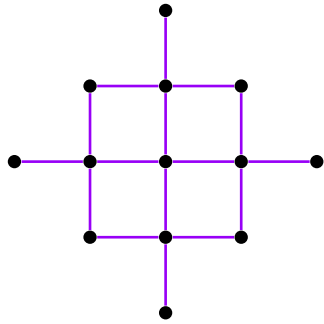
**Theorem 4.2.6.** For  $n \geq 6$ ,  $IR(B_n) = 4n - 14$



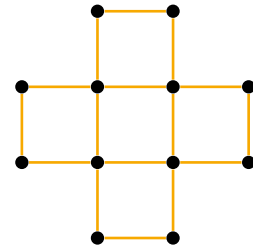
(a) A  $5 \times 5$  Bishop graph  $B_5$ .



(b)  $B_5$  redrawn as a Rook graph rotated 45 degrees.



(c)  $B_5$  black component rotated 45 degrees. Note the  $3 \times 3$  Rook graph inside indicating that the domination number of the black component is at least 3.



(d)  $B_5$  white component rotated 45 degrees. Note the  $2 \times 2$  Rook graph inside indicating that the domination number of the white component is at least 2.

Figure 4.2.1:  $B_5$  rotated and divided into its white and black components. The edges corresponding to moves between black squares are given in purple, moves between white squares are given in orange. Since the black and white graphs form separate components of  $B_5$ , we have  $\gamma(B_5) \geq 5$ .

### 4.3 King

Since the movement of a king forces that the piece can dominate at most  $3 \times 3$  section of the board at any one point in time, it is easier to determine those parameters involved in the domination chain. Specifically, for any  $n \times n$  chessboard, we know the dominating number, independent dominating number, and independence number for the associated  $n \times n$  King graph,  $K(n, n) = K_n$ . Dominating and independent sets for the  $5 \times 5$  King graph can be seen in Figure 4.3.1.

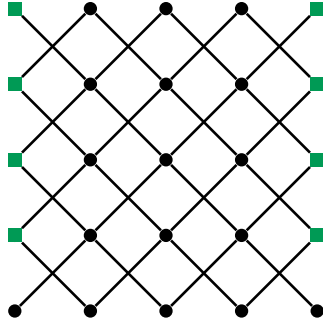
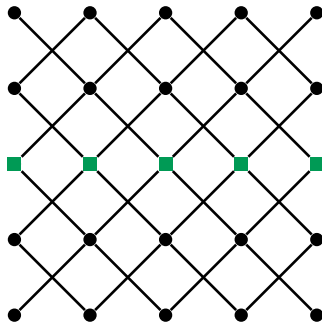
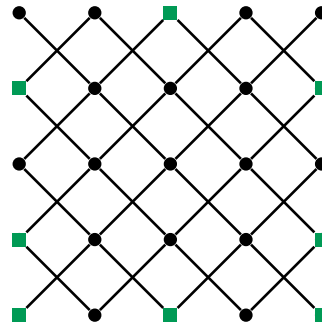


Figure 4.2.2:  $\alpha(B_5) = 8$  with independent set in green.



(a)  $i(B_5) = 5$  with independent dominating set in green on the middle row.



(b)  $\Gamma(B_5) = 8$  with upper dominating set in green along the border.

Figure 4.2.3:  $B_5$  independent dominating and upper dominating sets.

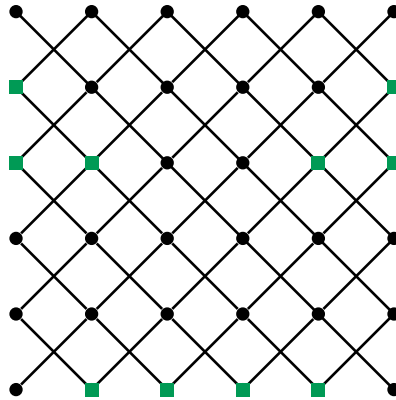
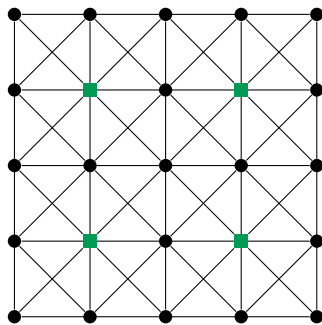
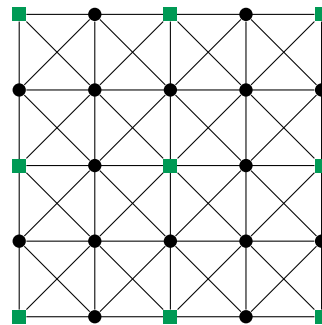
**Theorem 4.3.1.** ([2], [8])  $\gamma(K_n) = \lfloor \frac{n+2}{3} \rfloor^2$ .

**Corollary 4.3.2.** ([2], [8])  $i(K_n) = \lfloor \frac{n+2}{3} \rfloor^2$ .

**Theorem 4.3.3.** ([2], [8]) For  $n \geq 1$ ,  $\alpha(K_n) = \lfloor \frac{n+1}{2} \rfloor^2$ .

Kings domination and independence is easily extended to  $m \times n$  boards due to the finite nature of the king moves. Adjusting Theorem 4.3.1 and Corollary 4.3.2 for  $m$  and  $n$  yields  $\gamma(K(m, n)) = K(i(m, n)) = \lfloor \frac{m+2}{3} \rfloor \lfloor \frac{n+2}{3} \rfloor$ . Adjusting Theorem 4.3.3 similarly yields  $\alpha(K(m, n)) = \lfloor \frac{m+1}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor$ .

Unlike with rooks and bishops, the irredundance numbers for the King graph are not fully known. However, all known bounds for both the lower and upper irredundance numbers are due to Favoron et al. [7] as stated below. Figure 4.3.2 provides an

Figure 4.2.4:  $B_6$  with upper irredundance in green.(a)  $K_5$  with independent dominating set.(b)  $K_5$  with independent set.Figure 4.3.1:  $K_5$  with independent dominating set and independent set.

example of an irredundant set and Figure 4.3.1b gives an example of an upper irredundant set. Currently, there are no known results for the upper domination number for a king.

**Theorem 4.3.4.** ([7])  $ir(K_n) \leq \lfloor \frac{n+2}{3} \rfloor^2 - 1$  when  $n \equiv 4 \pmod{6}$ .

**Theorem 4.3.5.** ([7])  $\lceil \frac{n^2}{9} \rceil \leq ir(K_n) \leq \lfloor \frac{n+2}{3} \rfloor^2$  and so  $ir(K_n) = \frac{n^2}{9}$  when  $n \equiv 0 \pmod{3}$ .

**Theorem 4.3.6.** ([7]) For  $n \geq 6$ ,  $\frac{(n-1)^2}{3} \leq IR(K_n) \leq \frac{n^2}{3}$ .

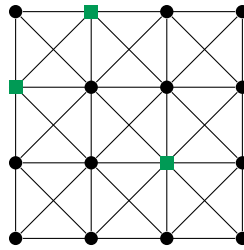


Figure 4.3.2:  $ir(K_4) = 3$  as per Theorem 4.3.4 with irredundant set in three of the center vertices.

## 4.4 Queen

One of the most well-known problems, a packing problem, first posed for an  $8 \times 8$  by chess puzzle composer Max Bezzel [8], is this: How many queens can be placed on a chessboard so that no queen attacks another? For an  $n \times n$  board the answer is  $n$  queens.

Thus, one parameter in our domination chain, the independence number, is known for  $Q(n, n) = Q_n$ . For the other five parameters, however, only bounds can be given, and much of what is known is due to computer searches.

**Theorem 4.4.1.** ([1]) For  $n > 3$ ,  $\alpha(Q_n) = n$ .

As in Corollary 4.1.3, we can generalize our queens independence to an  $m \times n$  board. We achieve for  $n > 3$ ,  $\alpha(Q(m, n)) = \min\{m, n\}$ .

Perhaps because the Queen graph problems are particularly challenging, it is well studied. Such questions that have been raised about this class of graphs include “What is the fewest number of queens needed to attack or occupy every square on the board?” (If we took out the words “or occupy” we would be looking for the total domination value for a given board). The answer to this posed question for an  $8 \times 8$  board, is exactly 5 queens, or  $\gamma(Q_8) = 5$ . However, there is no known generalization of this 5-queen result; this problem remains open for an  $n \times n$  board since a formula has yet to be determined for the number of queens needed. Nevertheless, some reasonable lower bounds for queens domination on a square board do exist.



**Theorem 4.4.2.** ([18])  $\gamma(Q_n) \geq \frac{1}{2}(n - 1)$ .

**Corollary 4.4.3.** ([18]) For  $n = 4k + 1$ ,  $\gamma(Q_n) \geq \frac{1}{2}(n + 1) = 2k + 1$ .

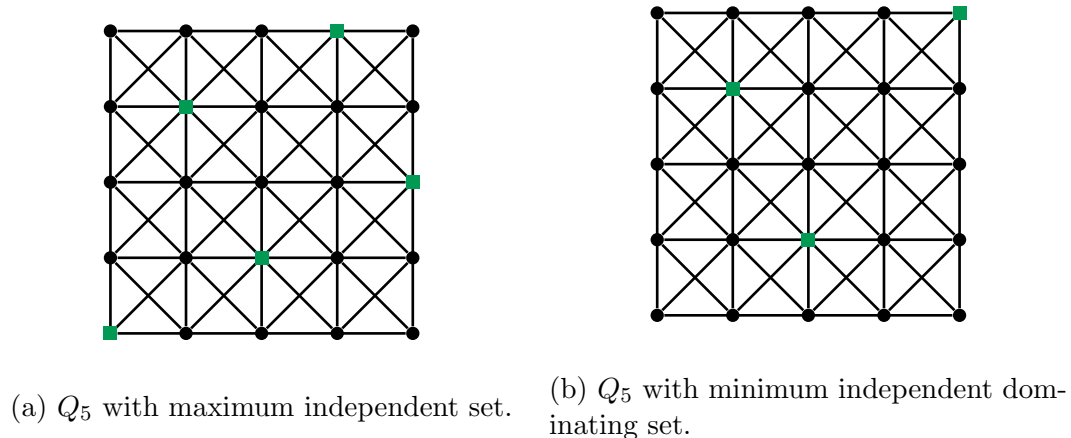


Figure 4.4.1:  $Q_5$  independent set and independent dominating set.

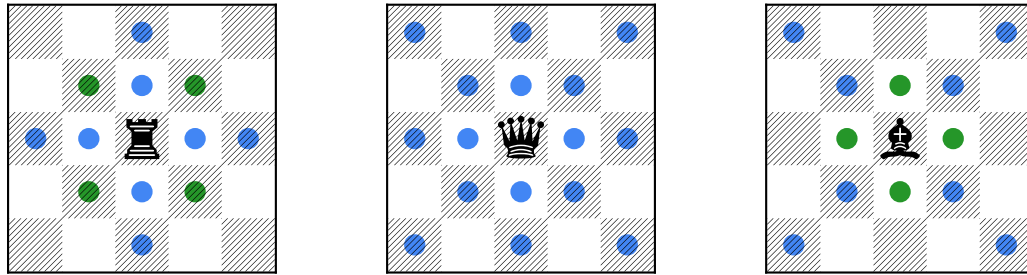
As stated in Chapter 2, the queen has been studied using other pieces from different variations of chess. These new pieces include the dragon king ( $Dk_n$ ) and dragon horse ( $Dh_n$ ) pieces from the Japanese chess game Shogi [3]. The dragon king and dragon horse combine the moves of the king with the rook and bishop respectively and, thus, the movement is closer to that of the queen than any other piece in classic chess. The goal is to use these different chess pieces to find bounds for both the domination number and independent domination number of their graphs and then use these results to gain greater insight into bounding these same parameters for the Queen graph. The moves for these pieces are in Figure 4.4.2 along with the movement for the Queen.

**Theorem 4.4.4.** ([3]) For  $n \geq 7$ ,  $\gamma(Dk_n) = i(Dk_n) = n - 3$ .

**Theorem 4.4.5.** ([3]) For  $n \geq 4$ ,  $\gamma(Dh_n) \leq n - 1$ .

**Conjecture 4.4.6.** ([8]) For  $n$  sufficiently large,  $\gamma(Q_n) = i(Q_n)$ .

**Theorem 4.4.7.** ([19]) For  $n \geq 5$ ,  $\Gamma(Q_n) \geq 2n - 5$ .



(a) Dragon King

(b) Classic Queen

(c) Dragon Horse

Figure 4.4.2: The dragon king has the combined moves of a rook and king, while the dragon horse has the combined moves of a bishop and king. These are most closely related to the queen's movement, as shown in (b). The original moves of the rook and bishop are in given in blue with the additional king moves in green.

$n$	1	2	3	4	5	6	7	8	9	10
$ir(Q_n)$	1	1	1	2						
$IR(Q_n)$	1	1	2	4	5	7	9	11		

Table 4.1: Irredundance values for  $n \times n$  Queen graph.

There are perhaps no good upper bounds for  $ir(Q_n)$  or  $IR(Q_n)$ . Several parameters have been computer calculated for various  $n \times n$  boards as given in Table 4.1 [12].

## 4.5 Knight

Since the Knight graph  $N(n, n) = N_n$  is bipartite,  $\alpha(N_n) = \Gamma(N_n) = IR(N_n)$  as proved by Cockayne et al. [4]. Recall from Chapter 2 that each successive knight move forces the knight to change to a square of opposing color. Thus, we can intuitively see that, by placing a knight on every square of one particular color, we can maximize our knight placement. If  $n$  is even, we can choose either of the two colors for this arrangement, while if  $n$  is odd we would choose that color with a larger number of squares.

**Theorem 4.5.1.** ([4])

$$\alpha(N_n) = \Gamma(N_n) = IR(N_n) = \begin{cases} \frac{n^2}{2} & \text{for } n \text{ even} \\ \frac{n^2+1}{2} & \text{for } n \text{ odd} \end{cases}$$

We also know the maximization parameters for the  $m \times n$  Knight graph by using  $mn$  instead of  $n^2$  and extending the result above:

$$\alpha(N(m, n)) = \Gamma(N(m, n)) = IR(N(m, n)) = \begin{cases} \frac{mn}{2} & \text{for } n \text{ even} \\ \frac{mn+1}{2} & \text{for } n \text{ odd} \end{cases}$$

In 1987 Hare and Hedetniemi [11] published a linear-time algorithm to find the domination number for a given Knight graph. This algorithm returns the domination number for  $m \times n$  boards.



(a)  $\alpha(N_4) = 8$  with independent set along the border in green. (b)  $\gamma(N_4) = 4$  with dominating set on the center vertices in green.

Figure 4.5.1:  $N_4$  independent set and dominating set.

Interestingly, the knight's  $n \times n$  irredundance number is studied only for  $n = 1, 2$ . Moreover, the first three parameters in the domination chain  $ir(N)$ ,  $\gamma(N)$ , and  $i(N)$  don't have any good upper bounds.  $ir(N_n)$  is virtually unstudied, while  $i(N_n)$  values are known for specific  $n$  as seen in Table 4.2 [12].

$n$	1	2	3	4	5	6	7	8	9	10
$ir(N_n)$	1	4								
$\gamma(N_n)$	1	4	4	4	5	8	10	12	14	16
$i(N_n)$	1	4	4	4	5	8	13	14	14	16

Table 4.2: Minimization parameter values for  $n \times n$  Knight graph.

# Chapter 5

## New Results

In this Chapter we explore our new domination chain results for the  $m \times n$  Bishop graph. Without loss of generality we will assume  $m \leq n$  since the  $m \times n$  Bishop graph is isomorphic to a Bishop graph with  $n$  rows and  $m$  columns.

**Definition 5.0.2.** A  $m \times n$  chessboard has  $m$  rows and  $n$  columns of alternating black and white squares. The bottom left square is black.

### 5.1 Bishop Movement on an $m \times n$ Graph

Consider the  $m \times n$  chessboard. We may denote each square as an ordered pair of whole numbers  $(x, y)$ ,  $0 \leq x \leq n - 1$ ,  $0 \leq y \leq m - 1$ . An  $m \times n$  board has  $m$  rows and  $n$  columns. Figure 5.1.1 provides an example.

**Proposition 5.1.1.** Let  $(x, y)$  be any square on the  $m \times n$  chessboard such that  $x$  and  $y$  are whole numbers,  $0 \leq x \leq n - 1$ ,  $0 \leq y \leq m - 1$ . For an integer  $k$ , a bishop placed at  $(x, y)$  may move to any square of the form:

$$(x + k, y + k) \text{ with } \max\{-y, -x\} \leq k \leq \min\{m - y - 1, n - x - 1\}, k \neq 0 \quad (5.1.1)$$

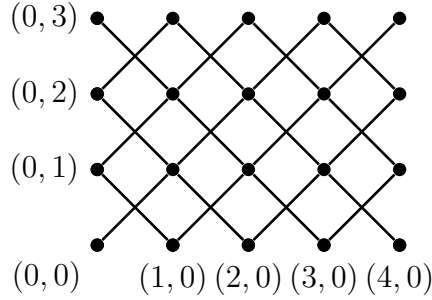


Figure 5.1.1: A  $4 \times 5$  simplified Bishop graph with four rows and five columns.

or

$$(x + k, y - k) \text{ with } \max\{-m + y + 1, -x\} \leq k \leq \min\{y, n - x - 1\}, k \neq 0 \quad (5.1.2)$$

*Proof.* From Section 2.3, a bishop can move to any square on the same two diagonals, the northeast to southwest diagonal, modeled by Equation (5.1.1), and the northwest to southeast, Equation (5.1.2). Due to this movement, we just need to show that the bounds given in these equations both contain the bishop on the board and contact an edge, thus showing that the movement described encompasses all possible squares on the two diagonals.

Let  $(x, y)$  be a square on the  $m \times n$  chessboard,  $0 \leq x \leq n - 1$ ,  $0 \leq y \leq m - 1$ . Let us consider the northeast to southwest diagonal.

**Case 1:** Consider Equation (5.1.1). We have  $\max\{-x, -y\} \leq k \leq \min\{m - y - 1, n - x - 1\}$ .

Consider the following subcases:

**Case 1a:** Let  $x \leq y$ . Then  $\max\{-y, -x\} = -x$  and our bound becomes  $-x \leq k \leq \min\{m - y - 1, n - x - 1\}$ ,  $k \neq 0$ . (Note: The lower bound becomes  $-x \leq k$ , regardless of whether  $m - y - 1$  or  $n - x - 1$  is the maximum.)

Using this lower bound we obtain  $(x - x, y - x)$ , which is  $(0, y - x)$ . Since  $x \geq 0$

and  $x \leq y$ ,  $(0, y - x)$  is clearly on the board and touches the left edge.

If  $m - y - 1 \leq n - x - 1$ , then  $-x \leq k \leq m - y - 1$  and the bishop can move to any square on the diagonal from  $(x - x, y - x)$  to  $(x + m - y - 1, y + m - y - 1)$ . That is, from  $(0, y - x)$  to  $(x + m - y - 1, m - 1)$ . Since  $x \geq 0$ ,  $m - y - 1 \geq 0$ , and  $m - y - 1 \leq n - x - 1$ , we obtain  $0 \leq x + m - y - 1 \leq n - 1$ . Thus  $(x + m - y - 1, m - 1)$  is on the board and touches the top edge. Therefore for  $-x \leq k \leq m - y - 1$ , Equation (5.1.1) holds.

If  $n - x - 1 \leq m - y - 1$ , then  $-x \leq k \leq n - x - 1$  and the bishop can move to any square on the diagonal from  $(x - x, y - x) = (0, y - x)$  to  $(x + n - x - 1, y + n - x - 1) = (n - 1, y + n - x - 1)$ . Now  $y \geq 0$ , so  $n - x - 1 \geq 0$ , and  $n - x - 1 \leq m - y - 1$ , and we obtain  $0 \leq y + n - x - 1 \leq n - 1$ . Thus,  $(n - 1, y + n - x - 1)$  is on the board and touches the right edge. Therefore for  $-x \leq k \leq n - x - 1$  and Equation (5.1.1) holds.

**Case 1b:** If  $x > y$ , then  $\max\{-y, -x\} = -y$  and the bound under consideration becomes  $-y \leq k \leq \min\{m - y - 1, n - x - 1\}$ ,  $k \neq 0$  (Note: The lower bound becomes  $-y \leq k$ , regardless of whether  $m - y - 1$  or  $n - x - 1$  is the maximum.)

Preceding above we achieve  $(x - y, y - y) = (x - y, 0)$ . Since  $y \geq 0$  and  $y \leq x$ ,  $(x - y, 0)$  is clearly on the board and touches the bottom edge.

If  $m - y - 1 \leq n - x - 1$ , then  $-y \leq k \leq m - y - 1$  and the bishop can move to any square on the diagonal from  $(x - y, y - y)$  to  $(x + m - y - 1, y + m - y - 1)$ , from  $(x - y, 0)$  to  $(x + m - y - 1, m - 1)$ . As before,  $(x + m - y - 1, m - 1)$  is a square on the top of the board. Therefore for  $-y \leq k \leq m - y - 1$ , Equation (5.1.1) holds.

If  $n - x - 1 \leq m - y - 1$ , then  $-y \leq k \leq n - x - 1$  and the bishop can move to any square on the diagonal from  $(x - y, y - y) = (x - y, 0)$  to  $(x + n - x - 1, y + n - x - 1) = (n - 1, y + n - x - 1)$ , this latter point which is located on the right edge of the board. Therefore for  $-y \leq k \leq n - x - 1$  and Equation (5.1.1) holds.

**Case 2:** In considering Equation (5.1.2) we have  $\max\{-m + y + 1, -x\} \leq k \leq \min\{y, n - x - 1\}$ .

Consider the following subcases:

**Case 2a:** If  $x \leq m - y - 1$ , then  $\max\{-m + y + 1, -x\} = -x$  and the bound becomes  $-x \leq k \leq \min\{y, n - x - 1\}$ ,  $k \neq 0$ . (Note: The lower bound becomes  $-x \leq k$ , regardless of whether  $-m + y + 1$  or  $n - x - 1$  is the maximum.)

With this lower bound, we arrive at  $(x - x, y + x) = (0, y + x)$ . Since  $x \geq 0$ ,  $y \geq 0$ , and  $x \leq m - y - 1$ , we get  $0 \leq x + y \leq y + m - y - 1 = m - 1$ . Then  $(0, y + x)$  is on the left edge of the board.

If  $y \leq n - x - 1$ , then  $-x \leq k \leq y$  and the bishop can move to any square on the diagonal from  $(x - x, y + x)$  to  $(x + y, y - y)$ , so from  $(0, y + x)$  to  $(x + y, 0)$ .

Now since,  $x + y \geq 0$  and  $y \leq n - x - 1$ , we get  $0 \leq x + y \leq x + n - x - 1 = n - 1$ . Thus  $(x + y, 0)$  is on the board and touches the bottom edge. Therefore for  $-x \leq k \leq y$ , Equation (5.1.2) holds.

If  $n - x - 1 \leq y$ , then  $-x \leq k \leq n - x - 1$  and the bishop can move to any square on the diagonal from  $(x - x, y + x)$  to  $(x + n - x - 1, y - n + x + 1)$ , that is from  $(0, y + x)$  to  $(n - 1, y - n + x + 1)$ .

As before,  $(0, y + x)$  is on the board and touches the left edge. Since  $y \geq n - x - 1$ ,  $y - (n - x - 1) = y - n + x + 1 \geq 0$ , and since  $m - 1 \geq y \geq n - x - 1$  we get  $y - (n - x - 1) = y - n + x + 1 \leq m - 1$ . Then  $(n - 1, y - n + x + 1)$  is on the board and touches the right edge. Therefore for  $-x \leq k \leq n - x - 1$ , Equation (5.1.2) holds.

**Case 2b:** Let  $m - y - 1 \leq x$ .

We have  $\max\{-m + y + 1, -x\} = -m + y + 1$  and the equality becomes  $-m + y + 1 \leq k \leq \min\{y, n - x - 1\}$ ,  $k \neq 0$ . (Note: The lower bound becomes  $-m + y + 1 \leq k$ , regardless of whether  $m - y - 1$  or  $n - x - 1$  is the maximum.)

Using the lower bound where  $k = -m + y + 1$ , we achieve  $(x - m + y + 1, y + m - y - 1)$ , or  $(x - m + y + 1, m - 1)$ . Since  $x \geq m - y - 1$ , we get  $x - m + y + 1 \geq 0$  and since  $x \leq n - 1$  we get  $x - (m - y - 1) = x - m + y + 1 \leq n - 1$ . Thus,  $(x - m + y + 1, m - 1)$  is on the board and touches the top edge of the board.



If  $y \leq n - x - 1$ , then  $-m + y + 1 \leq k \leq y$  and the bishop can move to any square on the diagonal from  $(x - m + y + 1, y + m - y - 1)$  to  $(x + y, y - y)$ , that is from  $(x - m + y + 1, m - 1)$  to  $(x + y, 0)$ . In case 2a  $(x + y, 0)$  was shown to be on the bottom of the board. Therefore for  $-m + y + 1 \leq k \leq y$ , Equation (5.1.2) holds.

If  $n - x - 1 \leq y$ , then  $-m + y + 1 \leq k \leq n - x - 1$  and the bishop can move to any square on the diagonal from  $(x - m + y + 1, y + m - y - 1)$  to  $(x + n - x - 1, y - n + x + 1)$ , that is from  $(x - m + y + 1, m - 1)$  to  $(n - 1, y - n + x + 1)$ . From case 2a,  $(n - 1, y - n + x + 1)$  is on the right edge of the board. Therefore for  $-m + y + 1 \leq k \leq n - x - 1$ , Equation (5.1.2) holds.

□

## 5.2 Bishop Graph Definitions

**Definition 5.2.1.** The *Bishop graph* for an  $m \times n$  chessboard,  $B(m, n)$ , has vertex set  $V(B(m, n)) = \{(x, y) \mid 0 \leq x \leq n - 1, 0 \leq y \leq m - 1\}$  with edge set  $E(B(m, n))$  containing all edges of the form  $(x, y) \leftrightarrow (w, z)$  where vertex  $(x, y)$  can move to vertex  $(w, z)$  as prescribed in Proposition 5.1.1.

Note: Unless otherwise stated, assume  $0 \leq x \leq n - 1$  and  $0 \leq y \leq m - 1$

**Definition 5.2.2.** For a vertex  $(x, y)$  of  $B(m, n)$ , a *positive diagonal* containing  $(x, y)$  also contains the set of vertices that satisfy Equation (5.1.1).

**Definition 5.2.3.** For a vertex  $(x, y)$  of  $B(m, n)$ , a *negative diagonal* containing  $(x, y)$  also contains the set of vertices that satisfy Equation (5.1.2).

**Definition 5.2.4.** The *positive diagonal graph* is the subgraph  $B^+(m, n)$  with vertex set  $V(B(m, n))$  and edge set consisting of those edges between any two vertices on the same positive diagonal.

**Definition 5.2.5.** The *negative diagonal graph* is the subgraph  $B^-(m, n)$  with vertex set  $V(B(m, n))$  and edge set consisting of those edges between any two vertices on the same negative diagonal.

Note: Every vertex lays on exactly one positive diagonal and one negative diagonal.

**Definition 5.2.6.** A graph is *positively covered by vertex set  $S$*  if  $S$  contains at least one vertex from each positive diagonal in the graph.

**Definition 5.2.7.** A graph is *negatively covered by vertex set  $S$*  if  $S$  contains at least one vertex from each negative diagonal in the graph.

**Definition 5.2.8.** The *origin*  $(0, 0)$  is the lower left square on the Bishop graph. The origin is a black square.

**Definition 5.2.9.** The *Black Bishop graph*  $B_b(m, n)$  is the component of the Bishop graph containing the origin. That is,  $B_b(m, n)$  is the induced subgraph of  $B(m, n)$  such that  $(x, y) \in V(B_b(m, n))$  if and only if  $x + y$  is even.

**Definition 5.2.10.** The *black positive diagonal graph* is the induced subgraph  $B_b^+(m, n)$  of  $B^+(m, n)$  where  $v \in V(B_b^+(m, n))$  if and only if  $v \in V(B_b(m, n))$ .

**Definition 5.2.11.** The *black negative diagonal graph* is the induced subgraph  $B_b^-(m, n)$  of  $B^-(m, n)$  where  $v \in V(B_b^-(m, n))$  if and only if  $v \in V(B_b(m, n))$ .

We observe that  $B_b(m, n)$  represents the range of motion on the black squares of an  $m \times n$  chessboard.  $B_b^+(m, n)$  and  $B_b^-(m, n)$  represent positively directed movement and negatively directed movement on the black squares, respectively.

**Definition 5.2.12.** The *White Bishop graph*,  $B_w(m, n)$ , is the component of the Bishop graph not containing the origin. That is,  $B_w(m, n)$  is the induced subgraph of  $B(m, n)$  such that  $(x, y) \in V(B_w(m, n))$  if and only if  $x + y$  is odd.

**Definition 5.2.13.** The *white positive diagonal graph* is the induced subgraph  $B_w^+(m, n)$  of  $B^+(m, n)$  where  $v \in V(B_w^+(m, n))$  if and only if  $v \in V(B_w(m, n))$ .

**Definition 5.2.14.** The *white negative diagonal graph* is the subgraph  $B_w^-(m, n)$  of  $B^-(m, n)$  where  $v \in V(B_w^-(m, n))$  if and only if  $v \in V(B_w(m, n))$ .

$B_w(m, n)$  represents the range of motion on the white squares of an  $m \times n$  chessboard.  $B_w^+(m, n)$  and  $B_w^-(m, n)$  represent positively directed movement and negatively directed movement on the white squares, respectively.

### 5.3 Independence of the $m \times n$ Bishop Graph

The purpose of this section is to establish a formula for the maximum number of bishops that can be placed on an  $m \times n$  board without attacking, that is, to determine the packing problem for  $m \times n$  bishops. Recall the definition of independence number and independent set. We prove a formula for the independence number of a Bishop graph,  $n \geq 2m$ , and give a construction to find an associated independent set.

Before we begin, it is important to count the number of positive and negative diagonals on the Bishop graph.

**Proposition 5.3.1.**  $B(m, n)$  has  $m + n - 1$  positive diagonals and  $m + n - 1$  negative diagonals.

*Proof.* Each positive (negative) diagonal begins from the left column or bottom (top) row. Because there are  $n$  columns and  $m$  rows and one corner vertex counted twice, we get  $n + m - 1$  positive (negative) diagonals.

□

Observe the positive diagonal graph  $B^+(3, 4)$  and negative diagonal graph  $B^-(3, 4)$  in Figure 5.3.1.



Figure 5.3.1:  $B^+(3,4)$  and  $B^-(3,4)$  each have  $3 + 4 - 1 = 6$  diagonals.

We will need to break our argument into three cases, all based on the parity of  $m$  and  $n$ :  $m$  odd,  $m$  even with subcases  $n$  odd and  $n$  even. First let us observe the differences in the black and white components of each of these types of graphs in Figure 5.3.2.

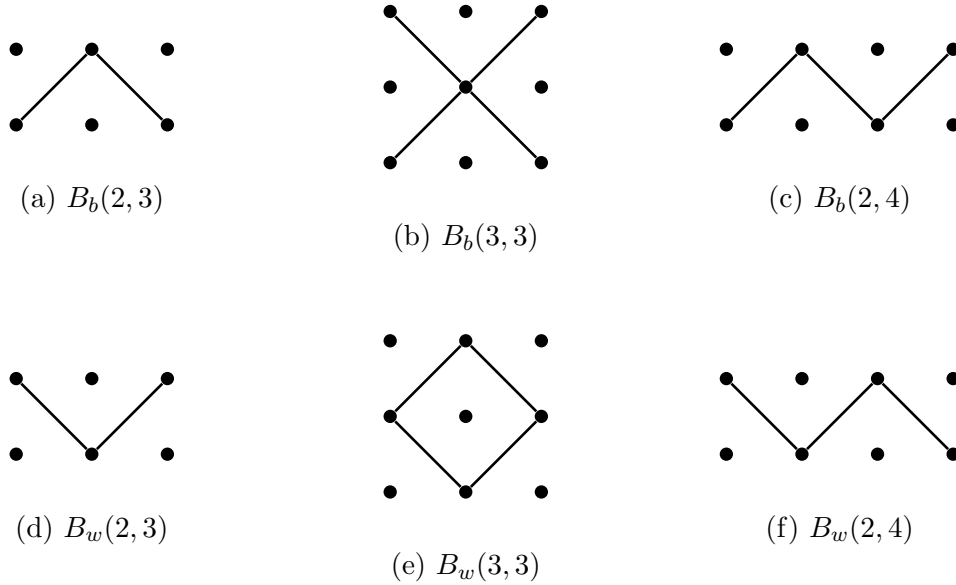


Figure 5.3.2: Black component graphs (top) with white component graphs (bottom).

In Figures 5.3.2a and 5.3.2d, we can see that the black and white component graphs  $B_b(2,3)$  and  $B_w(2,3)$  both have two positive and two negative diagonals. On the  $3 \times 3$  graphs in Figures 5.3.2b and 5.3.2e, we see that while the two component graphs are different, each component has the same number of positive and negative diagonals. That is  $B_b(3,3)$  has three positive and three negative diagonals and  $B_w(3,3)$  has two positive and two negative diagonals. In contrast, in Figure 5.3.2c,

$B_b(2, 4)$  has two positive and three negative diagonals while  $B_w(2, 4)$  in Figure 5.3.2f has three positive and two negative diagonals. This always occurs in the even  $\times$  even case, and affects the independence number for an even  $\times$  even graph.

**Proposition 5.3.2.** *If both  $m$  and  $n$  are even,  $B_b(m, n)$  has  $\frac{m+n}{2} - 1$  positive diagonals and  $\frac{m+n}{2}$  negative diagonals while  $B_w(m, n)$  has  $\frac{m+n}{2}$  positive diagonals and  $\frac{m+n}{2} - 1$  negative diagonals.*

*Proof.* Suppose both  $m$  and  $n$  are even. Consider  $B_b(m, n)$ . Since its diagonals lay on all vertices of the form  $(x, y)$  with  $x + y$  being even, we have a negative diagonal beginning at the origin and another negative diagonal ending on  $(n - 1, m - 1)$ . Since a vertex  $(x, y)$  on a row and column of opposite parity does not meet the criteria to be in  $B_b(m, n)$ , the vertices on  $B_b(m, n)$  alternate every other row along the first column beginning at the origin and every other column along the top row.. Thus we have  $\frac{m}{2} + \frac{n}{2} = \frac{m+n}{2}$  total negative diagonals (Figure 5.3.3c). We have a similar want for the positive diagonals, except that the diagonal emanating from the origin will be counted twice so we must subtract one for our final count of  $\frac{m+n}{2} - 1$  positive diagonals (Figure 5.3.3a). The case for  $B_w(m, n)$  gives the opposite result (Figures 5.3.3d and 5.3.3b).

□

**Proposition 5.3.3.** *A vertex  $(x, y)$  is on the same negative diagonal as a vertex  $(w, z)$  if  $x + y = w + z$ . Also, as we move left to right from one negative diagonal to the next, the sum increases.*

*Proof.* By Equation (5.1.2), our  $x$  increments by  $k$  and our  $y$  increments by  $-k$ , leaving a net change of 0 as we move between vertices on the same negative diagonal. Thus any two vertices on the same negative diagonal have the same sum for each ordered pair. As we move along the diagonals from vertices on the  $(0, 0)$  negative

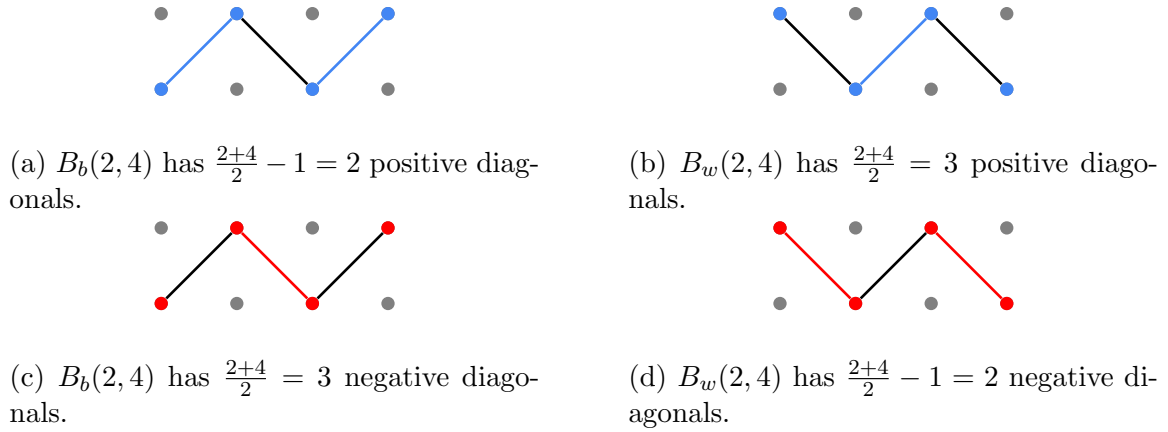


Figure 5.3.3: An even  $\times$  even example.

diagonal to vertices on the  $(n - 1, m - 1)$  diagonal, our sums increase from 0 to  $n - 1 + m - 1 = n + m - 2$  respectively. See  $B(3, 4)$  in Figure 5.3.4 for an example.  $\square$

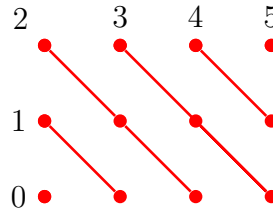


Figure 5.3.4: Every coordinate pair on a negative diagonal  $k$  for  $0 \leq k \leq 5$  has sum  $k$ .

Our argument makes heavy use of positive and negative diagonals, so we define some useful terminology here:

**Definition 5.3.4.** A vertex set on a graph  $G(m, n)$  is *positively independent* if it forms an independent set on  $G^+(m, n)$ .

**Definition 5.3.5.** A vertex set on a graph  $G(m, n)$  is *negatively independent* if it forms an independent set on  $G^-(m, n)$ .

**Theorem 5.3.6.** For the  $B(m, n)$  Bishop Graph:  $n \geq 2m$

$$\alpha(B(m, n)) = \begin{cases} m + n - 1 & \text{if at least one of } m, n \text{ odd} \\ m + n - 2 & \text{if both } m, n \text{ even} \end{cases}$$

*Proof. Case 1:* Let at least one of  $m, n$  be odd. From Proposition 5.3.1, since each bishop on a vertex attacks along the entire positive and negative diagonals per Equation (5.1.1) and Equation (5.1.2), respectively, we see that there can be at most  $m + n - 1$  non-attacking bishops on a  $B(m, n)$  graph. Thus we have an upper bound. In the two sub cases, we prove this is a lower bound by construction.

**Case 1a:** Assume  $m$  is odd. We will call our construction set  $S$ . We choose vertices on the left and right end columns  $S_L = (0, y)$  and  $S_R = (n - 1, y)$ ,  $0 \leq y \leq m - 1$ , and the center row  $S_M = (x, \frac{m-1}{2})$  where  $\frac{m+1}{2} \leq x \leq n - 1 - \frac{m+1}{2}$ . Then  $S = S_L + S_R + S_M$ .  $|S_L| = |S_R| = m$ , and  $|S_M| = n - 1 - \frac{m+1}{2} - \frac{m+1}{2} + 1 = n - m - 1$ . So  $|S| = |S_L| \cup |S_R| \cup |S_M| = m + m + n - m - 1 = n + m - 1$  as is necessary. Figure 5.3.5 provides an example.

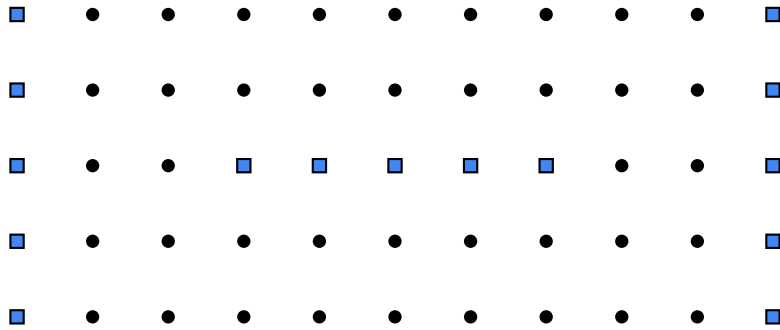


Figure 5.3.5: Independent set on a  $5 \times 11$  board.

It remains to be shown that set  $S$  is an independent set. It is clear from how the bishop moves that any set of vertices in the same column or same row are independent, as the bishop moves diagonally. Therefore the vertical sets  $S_L, S_R$ , each form an independent set and the horizontal set  $S_M$  forms an independent set and so we only need to check that  $S_L, S_R$ , and  $S_M$  are independent of each other.

Every positive diagonal emanating from  $S_L$  contains vertices of the form  $(x, y)$ ,  $x \leq$

$y$ . Since a vertex in  $S_M$  has the form  $(x, \frac{m-1}{2})$  with leftmost vertex  $(\frac{m+1}{2}, \frac{m-1}{2})$  clearly shows  $x > y$  with all subsequent  $x$ -coordinates increasing while the  $y$ -coordinate remains  $\frac{m-1}{2}$  we have that  $S_L$  and  $S_M$  are positively independent. Furthermore,  $S_R$  has the form  $(n-1, y)$  with  $y \leq m-1$  and  $n \geq 2m$  also giving  $x > y$ . Hence the bishops in  $S_L$  are positively independent from those in  $S_M$  and  $S_R$ . An example can be seen in Figure 5.3.6a.

By Proposition 5.3.3, the sum of any negative diagonal originating from a vertex  $(0, \ell)$  in  $S_L$ ,  $0 \leq \ell \leq m-1$ , is  $\ell$ . Therefore the sum of the coordinates of any negative diagonal from this column is less than or equal to  $m-1$ . Every vertex in  $S_M$  and  $S_R$  has a sum greater than or equal to  $m$ . Therefore  $S_L$  is negatively independent from both  $S_M$  and  $S_R$ . We can see this effect in Figure 5.3.6b. It follows that  $S_L$  is independent of  $S_M$  and  $S_R$ .



(a)  $S_L$  is positively independent from  $S_M$  and  $S_R$

(b)  $S_L$  is negatively independent from  $S_M$  and  $S_R$

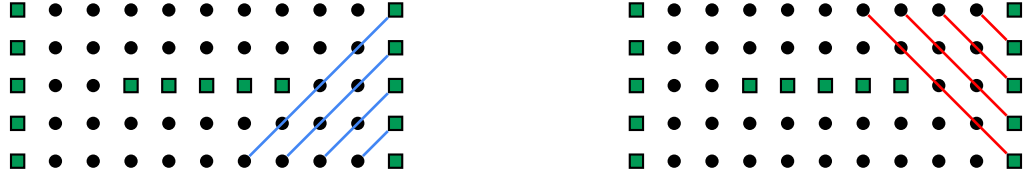
Figure 5.3.6:  $S_L$  is independent of  $S_M$  and  $S_R$ .

Turning our attention to  $S_R$ , we see that every positive diagonal emerging from a vertex  $(x, y)$  follows the rule  $x - n + m \geq y$ . All of vertices in  $S_M$  beginning from the right vertex  $(n-1 - \frac{m+1}{2}, \frac{m-1}{2})$  have  $x - n + m < y$  instead. Thus  $S_R$  and  $S_M$  are positively independent as in the example in Figure 5.3.7a.

Next, we again use Proposition 5.3.3 to see that sum of the negative diagonals of  $S_R$   $(n-1, \ell)$ ,  $0 \leq \ell \leq m-1$  is  $n-1+\ell$ . So the sum of the coordinates of any negative diagonal from this column is greater than or equal to  $n-1$ . Every vertex in  $S_M$  has sum less than or equal to  $n-2$ . Consequently,  $S_R$  is negatively independent from  $S_M$  as in Figure 5.3.7b. Therefore  $S_R$  and  $S_M$  form an independent set. As a result,  $S_L$ ,



$S_R$ , and  $S_M$  form an independent set  $S$  of size  $m + n - 1$  for the  $m$  odd case.



(a)  $S_R$  is positively independent from  $S_M$ .      (b)  $S_R$  is negatively independent from  $S_M$ .

Figure 5.3.7:  $S_R$  and  $S_M$  form an independent set.

**Case 1b:** Assume  $m$  is even and thus,  $n$  odd. We will call our construction set  $S$  as before. We choose vertices on the end columns  $S_L = (0, y)$  and  $S_R = (n - 1, y)$ ,  $0 \leq y \leq m - 1$  as before and vertices from the two center rows which we again call  $S_M$ . For  $S_M$  we choose top row  $(\frac{m}{2} + 1 + 2\ell, \frac{m}{2})$  and bottom row  $(\frac{m}{2} + 1 + 2\ell, \frac{m}{2} - 1)$  with  $0 \leq \ell \leq \frac{n-m-3}{2}$ . Thus we have  $S = S_L \cup S_R \cup S_M$ .  $|S_L| = |S_R| = m$  and  $|S_M| = 2(\frac{n-m-3}{2} + 1) = n - m - 3 + 2 = n - m - 1$ . Then  $|S| = |S_L| + |S_R| + |S_M| = m + m + n - m - 1 = m + n - 1$ . Once again we will show that  $S$  forms an independent set.

We need to check that each of  $S_L$ ,  $S_R$ , and  $S_M$  are independent from one another as before.

Every positive diagonal emanating from  $S_L$  contains vertices of the form  $(x, y)$ ,  $x \leq y$ . Since a vertex in any of the other columns has the form  $x > y$  we have  $S_L$  is positively independent with  $S_M$  and  $S_R$ .

As with Case 1a, Proposition 5.3.3 gives us that the sum of any negative diagonal originating from a vertex  $(0, \ell)$ ,  $0 \leq \ell \leq m - 1$ , is  $\ell$ . Therefore the sum of the coordinates of any negative diagonal starting from  $S_L$  is less than or equal to  $m - 1$ . Every vertex in  $S_M$  and  $S_R$  has a sum greater than or equal to  $m$ . Thus the  $S_L$  is negatively independent from  $S_M$  and  $S_R$ . Therefore the  $S_L$  forms an independent set with  $S_M$  and  $S_R$ . We can see this independence in Figure 5.3.8.

Unlike before, we must show that our center rows in  $S_M$  are independent from



(a)  $S_L$  is positively independent from  $S_M$  and  $S_R$ .

(b)  $S_L$  is negatively independent from  $S_M$  and  $S_R$ .

Figure 5.3.8:  $S_L$  is independent of  $S_M$  and  $S_R$ .

each other. We already know that the vertices on the top row are independent from each other and that the bottom row vertices are independent from each other. We also know that each vertical pair of vertices are independent. We only need to show that vertices not on the same row or column are independent. That is, a vertex on the top row need only be shown to be independent from vertices on the bottom row minus the vertex directly below it, and similarly for a bottom row vertex. Here we use that  $B_b(m, n)$  has an even ordered pair sum for any ordered pair  $(x, y)$  and  $B_w(m, n)$  has an odd ordered pair sum for any  $(x, y)$ . We already know that the black and white components of the graph are independent since the graphs are disconnected. Thus we will show that every vertex chosen on the top row is in the white component, and every vertex chosen on the bottom row is in the black component presenting to us that  $S_M$  forms an independent set. The top row vertices were stated to be of the form  $(x, \frac{m}{2})$  where  $x = \frac{m}{2} + 1 + 2\ell$  with  $0 \leq \ell \leq \frac{n-m-3}{2}$ . Since  $m$  is even,  $x$  is always odd by construction and since  $y$  is even, we get an odd coordinate sum. Therefore the top row belongs to  $B_w(m, n)$ . Now the bottom row has the form  $(x, \frac{m}{2} - 1)$  where  $x$  is still odd, but  $y$  is also odd this time, so we get an even coordinate sum, giving us that the bottom row belongs to  $B_b(m, n)$ . Therefore  $S_M$  forms an independent set.

As we once again turn our attention to the rightmost column  $S_R$ , we see that every positive diagonal originating from this column follows the rule  $x - n + m \geq y$ . All of our vertices in  $S_M$  beginning from the right  $(n - 2 - \frac{m}{2}, \frac{m}{2})$  and  $(n - 2 - \frac{m}{2}, \frac{m}{2} - 1)$  and moving to the left are shown to follow  $x - n + m < y$  instead. Thus  $S_M$  and  $S_R$

are positively independent.

We are left to show that  $S_M$  and  $S_R$  are negatively independent. We again use Proposition 5.3.3 to see that the negative diagonals of  $S_R$  ( $n - 1, \ell$ ),  $0 \leq \ell \leq m - 1$  have coordinate sum  $n - 1 + \ell$ . Thus the sum of the coordinates of any negative diagonal from this column is greater than or equal to  $n - 1$ . Every vertex on  $S_M$  has sum less than or equal to  $n - 2$ . Thus,  $S_M$  and  $S_R$  are negatively independent. Therefore  $S_M$  and  $S_R$  form an independent set. This can be seen in Figure 5.3.9. Hence we have an independent set  $S$  for the  $m$  even case.



(a)  $S_R$  is positively independent from  $S_M$ .      (b)  $S_R$  is negatively independent from  $S_M$ .

Figure 5.3.9:  $S_R$  and  $S_M$  form an independent set.

**Case 2:** Now consider when  $n$  and  $m$  are both even. By Proposition 5.3.2,  $B_b(m, n)$  has  $\frac{m+n}{2} - 1$  positive diagonals and  $\frac{m+n}{2}$  negative diagonals, whereas the opposite count is true for  $B_w(m, n)$ . So we can choose at most  $\frac{m+n}{2} - 1$  vertices from each component to get  $\frac{m+n}{2} - 1 + \frac{m+n}{2} - 1 = m + n - 2$  diagonals. To show that this is also the lower bound, we can give a construction.

We will call our construction  $S$ . We choose vertices on the end columns  $S_L = (0, y)$  and  $S_R = (n - 1, y)$ ,  $0 \leq y \leq m - 1$ , and the center row  $S_M = (x, \frac{m}{2})$  where  $\frac{m}{2} + 1 \leq x \leq n - 2 - \frac{m}{2}$ . Again,  $S = S_L \cup S_R \cup S_M$ .  $|S_L| = |S_R| = m$ , and  $|S_M| = n - 2 - \frac{m}{2} - (\frac{m}{2} + 1) + 1 = n - m - 2$  vertices from the row, giving a total of  $2m + n - m - 2 = n + m - 2$  vertices. Figures 5.3.10 and 5.3.11 provide an example.

The process for proving independence form an independent set is similar to Case 1a.

As we can see in Figure 5.3.10a, every positive diagonal emanating from the  $S_L$  contains vertices of the form  $(x, y)$ ,  $x \leq y$  as before. Since a vertex in  $S_M$  has the

form  $(x, \frac{m}{2})$  with  $\frac{m}{2} + 1 \leq x \leq n - 2 - \frac{m}{2}$  and  $S_R$  has form  $(n - 1, y)$ ,  $0 \leq y \leq m - 1$ , we have that  $S_L$  is positively independent from  $S_M$  and  $S_R$ .

As with Case 1a, by Proposition 5.3.3, the sum of any negative diagonal starting from a vertex in  $S_L$  is less than or equal to  $m - 1$ . Every vertex in  $S_M$  and  $S_R$  has a sum greater than or equal to  $m + 1$ . Therefore  $S_L$  is negatively independent from  $S_M$  and  $S_R$ . Thus the  $S_L$  forms an independent set with  $S_M$  and  $S_R$ . We can see this independence in Figure 5.3.10b.



(a)  $S_L$  positively independent from  $S_M$  and  $S_R$ .

(b)  $S_L$  is negatively independent from  $S_M$  and  $S_R$ .

Figure 5.3.10:  $S_L$  is independent of  $S_M$  and  $S_R$ .

Turning our attention to the  $S_R$ , we see again that every positive diagonal originating from this column follows the rule  $x - n + m \geq y$ . All of our vertices in  $S_M$  beginning from the right vertex  $(n - 2 - \frac{m}{2}, \frac{m}{2})$  and moving left are shown to follow  $x - n + m < y$  instead. Then  $S_M$  is positively independent from  $S_R$ .

Again using Proposition 5.3.3 we see that the sum of the negative diagonals of  $S_R$ ,  $(n - 1, \ell)$ ,  $0 \leq \ell \leq m - 1$  is  $n - 1 + \ell$ . So the sum of the coordinates of any negative diagonal from  $S_R$  is greater than or equal to  $n - 1$ . Every vertex in  $S_M$  has diagonal sum less than or equal to  $n - 2$ . Therefore, the  $S_R$  is negatively independent from  $S_M$ . Hence  $S_M$  and  $S_R$  form an independent set.

Consequently, we have an independent set for the even  $\times$  even case.

This completes the proof of the theorem.

□

Through a computer search we now present the values of  $\alpha(B(m, n))$  for  $m, n \leq 10$  in Table 5.1.



(a)  $S_R$  is positively independent from  $S_M$ .      (b)  $S_R$  is negatively independent from  $S_M$ .

Figure 5.3.11:  $S_L$ ,  $S_M$ , and  $S_R$  form an independent set.

$m \backslash n$	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	2	4	4	6	6	8	8	10	10
3	3	4	4	6	7	8	9	10	11	12
4	4	4	6	6	8	8	10	10	12	12
5	5	6	7	8	8	10	11	12	13	14
6	6	6	8	8	10	10	12	12	14	14
7	7	8	9	10	11	12	12	14	15	16
8	8	8	10	10	12	12	14	14	16	16
9	9	10	11	12	13	14	15	16	16	18
10	10	10	12	12	14	14	16	16	18	18

Table 5.1: Independence numbers for the  $m \times n$  Bishop graph.

## 5.4 Domination of the $m \times n$ Bishop Graph

We now present an upper bound for the minimum number of bishops that can cover an  $m \times n$  board for  $m \leq n \leq 2m$ . To do this, we provide a general construction for a dominating set for such a  $m \times n$  board. We also deliver a conjecture on the domination number for a  $m \times n$  Bishop graph. To begin, we deliver a short proposition.

**Proposition 5.4.1.**  $\gamma(B(m, n)) = n$  if  $m = 1$  or  $m = n$ .

*Proof.* We have two cases: Case 1: Let  $m = 1$ . Our graph is composed of one row of isolated vertices. Thus,  $\gamma(B(1, n))$  must be the total number of vertices,  $n$ .

Case 2: Let  $m = n$ . This result follows from Theorem 4.2.2 in Chapter 4.  $\square$

**Proposition 5.4.2.** For each positive diagonal  $p$  in  $B(m, n)$ , the vertices in  $p$  all have the same difference between their  $x$  and  $y$  coordinates. Moreover, as we move

left to right from one positive diagonal to the next, the difference values associated with these diagonals increase by one at each step and range from  $1 - m$  to  $n - 1$ .

*Proof.* Let  $p$  be a positive diagonal in  $B(m, n)$  and let  $(x, y)$  be a vertex of  $p$ . Clearly, the difference of the coordinates for this vertex is  $x - y$ . By Equation (5.1.1),  $p$  contains any vertex of the form  $(x + k, y + k)$ , so the  $x$ -coordinate increments by  $k$  as does the  $y$ -coordinate. Therefore the difference in the coordinates is  $x + k - (y + k) = x - y$  for any vertex along the same positive diagonal as  $(x, y)$  and any two vertices on the same positive diagonal  $p$  have the same difference of their coordinates.

Now, as we move along the diagonals from left to right, we have the following sequence of vertices as prescribed by the left and bottom edges of the board:  $(0, m - 1), (0, m - 2), \dots, (0, 1), (0, 0), (1, 0), \dots, (n - 1, 0)$ . Each of these vertices is on a different diagonal and the difference between any two consecutive entries in the sequence is one, beginning with  $0 - (m - 1) = 1 - m$  and ending with  $(n - 1) - 0 = n - 1$ .

□

We will be providing a construction that is also an independent dominating set.

**Theorem 5.4.3.**  $\gamma(B(m, n)) \leq 2 \lceil \frac{n-1}{2} \rceil$  if  $m < n \leq 2m$  or  $m \geq 2$ .

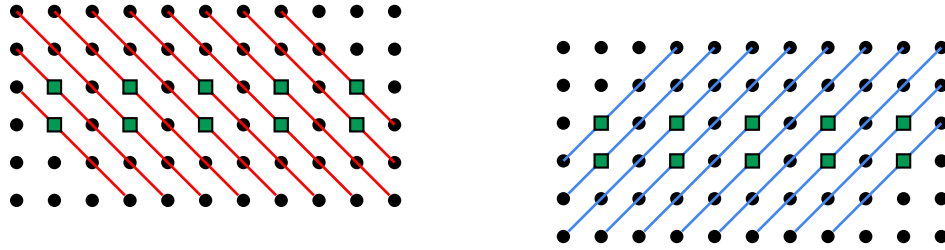
*Proof.* Let  $m < n \leq 2m$ . Consider the set  $S = S^U \cup S^D$  with  $S^U = \{(1 + 2\ell, \lfloor \frac{m}{2} \rfloor) \mid 0 \leq \ell \leq \lfloor \frac{n}{2} \rfloor - 1\}$  and  $S^D = \{(1 + 2\ell, \lfloor \frac{m}{2} \rfloor - 1) \mid 0 \leq \ell \leq \lfloor \frac{n}{2} \rfloor - 1\}$ . We have  $|S| = 2(\lfloor \frac{n}{2} \rfloor - 1 - 0 + 1) = 2 \lfloor \frac{n}{2} \rfloor = 2 \lceil \frac{n-1}{2} \rceil$ .

We need to show that set  $S$  dominates  $B(m, n)$ . First, let us determine those vertices that are negatively covered by  $S$ . The sums of the coordinates of the vertices in  $S$  include all values between  $\lfloor \frac{m}{2} \rfloor$  to  $2 \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor - 1$  so by Proposition 5.3.3, any vertex with a sum in this range is covered. We must now show those vertices not covered negatively are, in fact, covered by a positive diagonal. Since  $m < n$ , we have these uncovered vertices can be partitioned into two sets,  $L$  and  $R$ . (See Figure 5.4.1a for example).

Consider  $L$ .  $L$  consists of those vertices with coordinate sum  $s$ ,  $0 \leq s \leq \lfloor \frac{m}{2} \rfloor - 1$ . These vertices have, as differences of their coordinates, values that range between  $-\left(\lfloor \frac{m}{2} \rfloor - 1\right) = 1 - \lfloor \frac{m}{2} \rfloor$  and  $\lfloor \frac{m}{2} \rfloor - 1$ . Note that the values for the differences of the coordinates of those vertices in  $S$  range from  $1 - \lfloor \frac{m}{2} \rfloor$  to  $2 \lfloor \frac{n}{2} \rfloor - \lfloor \frac{m}{2} \rfloor$ . Since  $m < n$ , by Proposition 5.4.2 all vertices of  $L$  are positively covered.

Consider  $R$ .  $R$  consists of those vertices with coordinate sum  $s$ ,  $\lceil \frac{m}{2} \rceil + n - 1 = m + n - \lfloor \frac{m}{2} \rfloor - 1 \leq s \leq m + n - 2$ . The differences in the coordinates of these vertices in  $R$  are between  $n - \lfloor \frac{3m}{2} \rfloor + 2$  and  $n - \lceil \frac{m}{2} \rceil - 1$ , inclusive. Again using the coordinate differences of  $S$  and Proposition 5.4.2, we have each vertex in  $R$  is positively covered by some vertex in  $S$ .

Hence,  $S$  covers every vertex and is a dominating set. □



(a)  $L$  and  $R$  are the negatively uncovered vertices in the bottom left and upper right of the graph respectively.

(b)  $L$  and  $R$  are covered positively.

Figure 5.4.1:  $S$  is a dominating set.

**Lemma 5.4.4.** *The  $i^{\text{th}}$  column of a Bishop graph  $B(m, n)$ , that is, a set  $(i, y)$ ,  $0 \leq i \leq n - 1$ , dominates one  $m \times \lfloor \frac{m}{2} \rfloor$  set of vertices on either side and itself, for a total of one  $m \times (2 \lfloor \frac{m}{2} \rfloor + 1)$  set of vertices.*

*Proof.* Let  $C$  be the set of vertices on column  $i$ :  $C = \{(i, y) \mid 0 \leq y \leq m - 1\}$ , where  $0 \leq i \leq n - 1$ .

Now, any vertex  $(i, y)$  in  $C$  has sum  $i + y$ , so the range of the sums for all vertices in  $C$  is between  $i$  and  $i + m - 1$ . Thus, by Proposition 5.3.3, we have

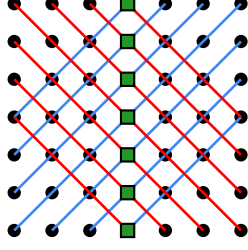
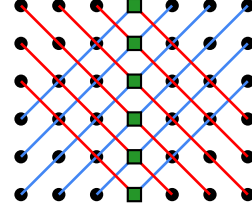
(a) An  $m$  is odd example.(b) An  $m$  is even example.

Figure 5.4.2: The 3<sup>rd</sup> column of  $B(7, 7)$  dominates a  $7 \times 2 \lfloor \frac{7}{2} \rfloor + 1 = 7$  rectangle and the 3<sup>rd</sup> column of  $B(6, 6)$  dominates a  $6 \times 2 \lfloor \frac{6}{2} \rfloor + 1 = 7$  rectangle.

that a vertex on the  $i^{\text{th}}$  column negatively covers any vertex  $(x, y)$  with sum  $x + y$  where  $i \leq x + y \leq i + m - 1$ . Similarly, any vertex  $(i, y) \in C$  has difference  $i - y$  and these values range from  $i - m + 1$  to  $i$ , inclusive, and we have (by again applying Proposition 5.4.2) some vertex on the  $i^{\text{th}}$  column positively covers any vertex with difference  $x - y$  with  $i - m + 1 \leq x - y \leq i$ .

We claim that  $C$  dominates the  $i - \lfloor \frac{m}{2} \rfloor^{\text{th}}$  column,  $L$ . We show this is true by proving  $L$  is covered from  $(i - \lfloor \frac{m}{2} \rfloor, 0)$  to  $(i - \lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor - 1)$  positively and from  $(i - \lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor)$  to  $(i - \lfloor \frac{m}{2} \rfloor, m - 1)$  negatively.

Consider the vertex  $(i - \lfloor \frac{m}{2} \rfloor, 0)$ . The difference for this coordinate is  $i - \lfloor \frac{m}{2} \rfloor$  and it is in the  $i - \lfloor \frac{m}{2} \rfloor^{\text{th}}$  column. From Proposition 5.4.2, we have  $i - m + 1 \leq i - \lfloor \frac{m}{2} \rfloor \leq i$ , and  $(i - \lfloor \frac{m}{2} \rfloor, 0)$  is covered positively by  $C$ . Similarly, vertex  $(i - \lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor - 1)$  has difference  $i - 2 \lfloor \frac{m}{2} \rfloor - 1$  and  $i - m + 1 \leq i - 2 \lfloor \frac{m}{2} \rfloor - 1 \leq i$ . Every vertex between these two vertices in  $L$  has a difference between their two differences. Thus, the lower portion of  $L$  is covered by a positive diagonal from a vertex in  $C$ .

Now consider the sum for the vertex with coordinate  $(i - \lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor)$ ,  $i - \lfloor \frac{m}{2} \rfloor + \lfloor \frac{m}{2} \rfloor = i$ . We can see this satisfies Proposition 5.3.3 as  $i \leq i - \lfloor \frac{m}{2} \rfloor + \lfloor \frac{m}{2} \rfloor \leq i + m - 1$ . Similarly, looking at the top vertex of  $L$ ,  $(i - \lfloor \frac{m}{2} \rfloor, m - 1)$ , we have sum  $i - \lfloor \frac{m}{2} \rfloor + m - 1$ . We can see that this satisfies Proposition 5.3.3 as  $i \leq i - \lfloor \frac{m}{2} \rfloor + m - 1 \leq i + m - 1$ . Thus, all vertices from  $(i - \lfloor \frac{m}{2} \rfloor, m - 1 - \lfloor \frac{m}{2} \rfloor)$  to  $(i - \lfloor \frac{m}{2} \rfloor, m - 1)$  are covered negatively



by some vertex in  $C$ . Therefore, the  $i - \lfloor \frac{m}{2} \rfloor^{\text{th}}$  column is dominated.

We will now also show that  $C$  dominates the  $i + \lfloor \frac{m}{2} \rfloor^{\text{th}}$  column,  $R$ . We will show this in a similar manner to how  $L$ 's domination by  $C$  was shown.

Now consider the sum for vertex  $(i - \lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor)$ ,  $i - \lfloor \frac{m}{2} \rfloor + \lfloor \frac{m}{2} \rfloor = i$ . We can see this satisfies Propostion 5.3.3 as  $i \leq i \leq i + m - 1$ . Similarly, looking at the topmost vertex in  $R$ ,  $(i - \lfloor \frac{m}{2} \rfloor, m - 1)$ , we have sum of  $i - \lfloor \frac{m}{2} \rfloor + m - 1$ , which is maximal among all vertices of  $R$ . We can see that this satisfies Propostion 5.3.3 as  $i \leq i - \lfloor \frac{m}{2} \rfloor + m - 1 \leq i + m - 1$ . Thus  $(i - \lfloor \frac{m}{2} \rfloor, m - 1 - \lfloor \frac{m}{2} \rfloor)$  to  $(i - \lfloor \frac{m}{2} \rfloor, m - 1)$  is covered negatively. Therefore, the  $i - \lfloor \frac{m}{2} \rfloor^{\text{th}}$  column is dominated.

Consider the difference for the vertex with coordinates  $(i + \lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor)$ ,  $i + \lfloor \frac{m}{2} \rfloor - \lfloor \frac{m}{2} \rfloor = i$ , on the  $i + \lfloor \frac{m}{2} \rfloor^{\text{th}}$  column. We can see this satisfies Proposition 5.4.2 as  $i - m + 1 \leq i \leq i$ . Similarly, looking at the top vertex of  $R$   $(i + \lfloor \frac{m}{2} \rfloor, m - 1)$ , we have difference  $i + \lfloor \frac{m}{2} \rfloor - m + 1 = i - \lceil \frac{m}{2} \rceil + 1$ . We can see that this satisfies Propostion 5.4.2 as  $i - m + 1 \leq i - \lceil \frac{m}{2} \rceil + 1 \leq i$ . Thus the upper half of  $R$ , vertices  $(i + \lfloor \frac{m}{2} \rfloor, m - 1 - \lfloor \frac{m}{2} \rfloor)$  to  $(i + \lfloor \frac{m}{2} \rfloor, m - 1)$ , is covered positively.

Looking at vertices  $(i + \lfloor \frac{m}{2} \rfloor, 0)$  and  $(i + \lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor - 1)$ , we have sums  $i + \lfloor \frac{m}{2} \rfloor$  and  $i + 2 \lfloor \frac{m}{2} \rfloor - 1$ , respectively. Again, it follows from Proposition 5.3.3 that these two vertices as well as all vertices in  $R$  between them are negatively covered by some vertex in  $C$ . Therefore, the  $i + \lfloor \frac{m}{2} \rfloor^{\text{th}}$  column is dominated.

The structure of the graph then yields that all columns between these  $L$  and  $R$  must also be covered as their sums and differences will fall in between the sums and differences of these two columns of vertices. Hence, the set of vertices from  $i - \lfloor \frac{m}{2} \rfloor^{\text{th}}$  to  $i - \lfloor \frac{m}{2} \rfloor^{\text{th}}$  is dominated. That is an  $m \times (2 \lfloor \frac{m}{2} \rfloor + 1)$  set of vertices is dominated.

□

We introduce the trapezoidal/triangular Bishop graph,  $B_T(m, n)$ .

**Definition 5.4.5.**  $B_T(m, n)$  is the induced subgraph of  $B(m, n)$  with  $m$  rows and  $n$  columns and whose vertex set consists of all vertices whose coordinate sum is  $s$ ,

$$0 \leq s \leq n - 1.$$

Note that when  $m = n$ , we can view  $B_T(m, n)$  as a right triangular graph, while for  $n > m$ , the board is a trapezoid, or “truncated” triangle.

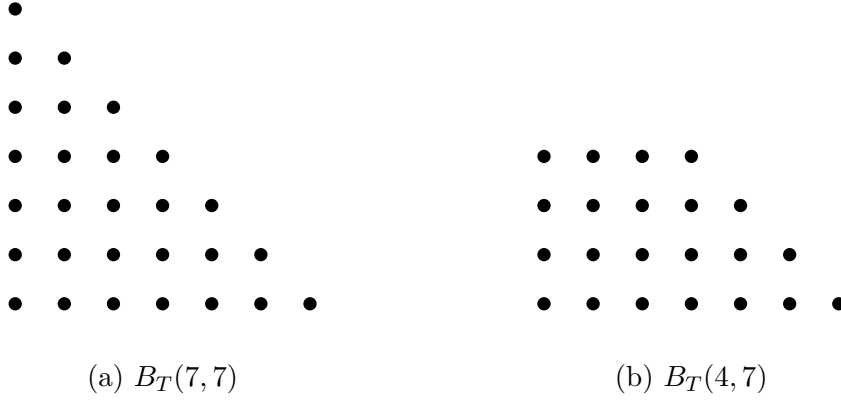


Figure 5.4.3: A triangular graph and trapezoidal graph.

**Theorem 5.4.6.** For  $m \geq 1$ ,  $i(B_T(m, m)) = \gamma(B_T(m, m)) = \lceil \frac{2m}{3} \rceil$ .

*Proof.* Let us consider two cases, based on the parity of the number of rows, mod 3:

If  $m \bmod 3 \equiv 1$  (the parity is odd): By Theorem 4.1.2, we need at least  $\lceil \frac{2m}{3} \rceil$  bishops since we have a  $\lceil \frac{2m}{3} \rceil \times \lceil \frac{2m}{3} \rceil$  Rook graph embedded in our triangular graph with corner vertices at  $(\lfloor \frac{m}{3} \rfloor, 0)$ ,  $(0, \lfloor \frac{m}{3} \rfloor)$ ,  $(2 \lfloor \frac{m}{3} \rfloor, \lfloor \frac{m}{3} \rfloor)$ , and  $(\lfloor \frac{m}{3} \rfloor, 2 \lfloor \frac{m}{3} \rfloor)$ . We need no more bishops than this because we may choose  $\lceil \frac{2m}{3} \rceil$  vertices along the  $(\lfloor \frac{m}{3} \rfloor, \ell)$  column of the triangular Bishop graph. That is, for  $0 \leq \ell \leq \lceil \frac{2m}{3} \rceil$ . Clearly, we have  $\lceil \frac{2m}{3} \rceil$  vertices in our set. They dominate a  $\lceil \frac{2m}{3} \rceil \times \lceil \frac{2m}{3} \rceil + 1$  rectangle by of Lemma 5.4.4. Notice there are vertices not in this rectangle on the top left and bottom right corners of our triangle. These are also covered due to sharing a negative diagonal with the selected set of vertices as confirmed by looking at the negative diagonal sums. Our top chosen vertex has sum  $\lfloor \frac{m}{3} \rfloor + \lceil \frac{2m}{3} \rceil = m$  while our bottom vertex and top vertex both lie on the same negative diagonal and have sum  $0 + m = m$ . Therefore, our vertex set covers the top vertex and bottom vertex. Since as we move

down the column, we also move down the set of negative diagonals, we will cover every vertex outside of our rectangle in this way.

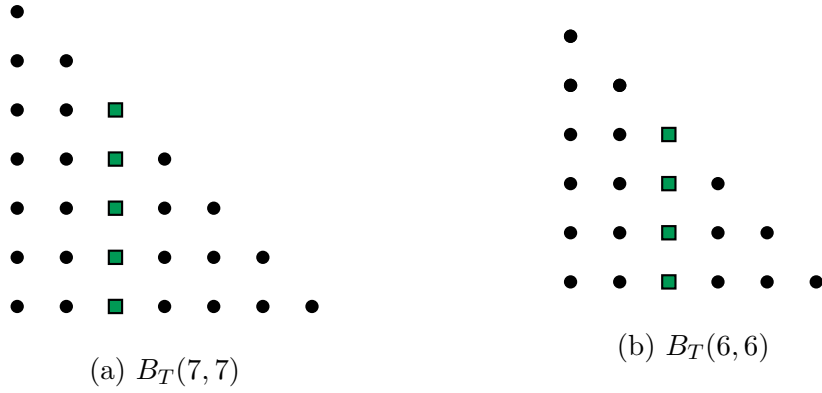


Figure 5.4.4:  $\gamma(B_T(7,7)) = \lceil \frac{2(7)}{3} \rceil = 5$  and  $\gamma(B_T(6,6)) = \lceil \frac{2(6)}{3} \rceil = 4$ .

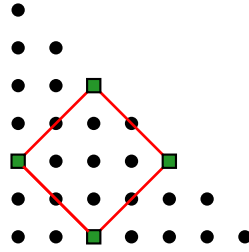


Figure 5.4.5: A  $5 \times 5$  Rook graph embedded yields at least  $\lceil \frac{2(7)}{3} \rceil = 5$  bishops.

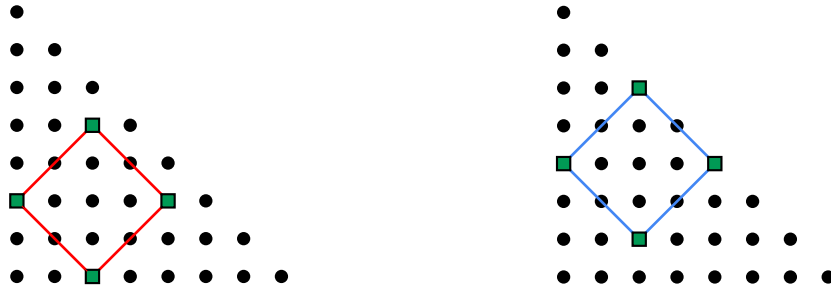
Now consider the case with  $m \bmod 3$  even: We need to view the black and white components of the board separately. When  $\lceil \frac{m}{3} \rceil$  is odd, our black component has an embedded  $\lceil \frac{m}{3} \rceil \times \lceil \frac{m}{3} \rceil$  Rook graph with corners  $(\lceil \frac{m}{3} \rceil - 1, 0)$ ,  $(0, \lceil \frac{m}{3} \rceil - 1)$ ,  $(2(\lceil \frac{m}{3} \rceil - 1), \lceil \frac{m}{3} \rceil - 1)$ , and  $(\lceil \frac{m}{3} \rceil - 1, 2(\lceil \frac{m}{3} \rceil - 1))$ . Our white component also has a  $\lceil \frac{m}{3} \rceil \times \lceil \frac{m}{3} \rceil$  Rook graph at  $(\lceil \frac{m}{3} \rceil - 1, 1)$ ,  $(0, \lceil \frac{m}{3} \rceil)$ ,  $(2(\lceil \frac{m}{3} \rceil - 1), \lceil \frac{m}{3} \rceil)$ , and  $(\lceil \frac{m}{3} \rceil - 1, (\lceil \frac{m}{3} \rceil))$ . Each component then, by Theorem 4.1.2, requires  $\lceil \frac{m}{3} \rceil$  bishops giving us  $\lceil \frac{2m}{3} \rceil$  total bishops needed to dominate the graph.

When  $\lceil \frac{m}{3} \rceil$  is even, the black and white component Rook graphs are interchanged.

We need no more bishops than this as we may choose  $\lceil \frac{2m}{3} \rceil$  vertices along the  $(\lceil \frac{m}{3} \rceil - 1, \ell)$  column of the triangular Bishop graph. That is for  $0 \leq \ell \leq \lceil \frac{2m}{3} \rceil$ .

Clearly, we have  $\lceil \frac{2m}{3} \rceil$  vertices in our set. These vertices form a dominating set for a  $\lceil \frac{2m}{3} \rceil \times \lceil \frac{2m}{3} \rceil + 1$  as seen in Lemma 5.4.4. Similar to case 1, this set also covers those vertices not in the rectangle.

Since these vertices are chosen in a vertical line, we have that they form an independent set. Thus,  $i(B_T(m, m)) = \gamma(B_T(m, m)) = \lceil \frac{2m}{3} \rceil$  for  $m > 1$ .



(a) Black component for  $\lceil \frac{m}{3} \rceil$  is odd case. (b) White component for  $\lceil \frac{m}{3} \rceil$  is odd case.

Figure 5.4.6: The embedded Rook graphs yield at least  $3 + 3 = \lceil \frac{2(8)}{3} \rceil = 6$  bishops.



(a) Black component for  $\lceil \frac{m}{3} \rceil$  is even case. (b) White component for  $\lceil \frac{m}{3} \rceil$  is even case.

Figure 5.4.7: The embedded Rook graphs yield at least  $2 + 2 = \lceil \frac{2(6)}{3} \rceil = 4$  bishops.

□

**Corollary 5.4.7.** For  $\lceil \frac{n}{3} \rceil \leq m \leq n$ ,  $\gamma(B_T(m, n)) \leq \lceil \frac{2n}{3} \rceil$ .

*Proof.* Consider  $B_T(n, n)$ . Choose the  $r^{\text{th}}$  row from the base of the  $n$  by  $n$  triangle where  $r = \lceil \frac{n}{3} \rceil$  when  $n \bmod 3 \equiv 1$  and  $r = \lceil \frac{n}{3} \rceil - 1$  when  $n \bmod 3$  is even. (Note that this is equivalent to picking the a column as in the proof of Theorem 5.4.6.)

Since these vertices are all in  $B_T(m, n)$  and dominate  $B_T(n, n)$  they also dominate  $B(T_m, n)$ .  $\square$

**Conjecture 5.4.8.**  $\gamma(B(m, n)) \leq 2 \lfloor \frac{m+n}{3} \rfloor$  if  $n > 2m$ .

Consider the following constructions for Conjecture 5.4.8.

Let  $n \geq 2m$ . Consider the set of vertices  $S = S_B \cup S_W$ , where  $S_X = S_X^I \cup S_X^U \cup S_X^D \cup S_X^T$ ,  $X \in \{B, W\}$ . Define:

- $S_B^I = \{(1, 1)\}$
- $S_B^D = \{(4 + 2i + j^D(6m - 6), 0) \mid 0 \leq i \leq m - 2, 0 \leq j^D \leq \lfloor \frac{n-2m-1}{6m-6} \rfloor\}$
- $S_B^U = \{(3m + 1 + 2i + j^U(6m - 6), m - 1) \mid 0 \leq i \leq m - 2, 0 \leq j^U \leq \lfloor \frac{n-5m+2}{6m-6} \rfloor\}$
- $S_B^T$  consists of those vertices chosen as in Theorem 5.4.6 or Corollary 5.4.7 where the triangle/trapezoid  $T$  has base of size  $b = (n - 2m - 1) \bmod (6m - 6)$  and height  $m$  and covers those vertices at the end of the board. Note, this base is along the bottom row of the board if  $j^D \neq j^U$  but along the top of the board for  $j^D = j^U$ . An example is provided in Figure 5.4.8.

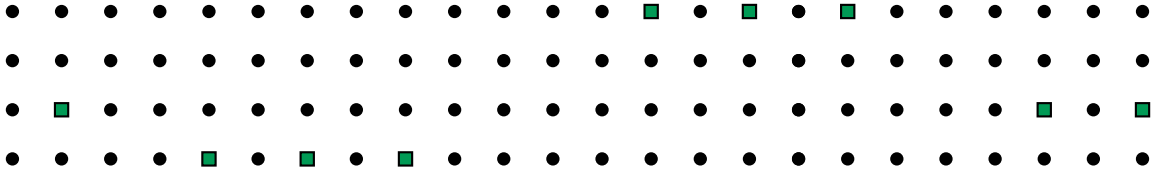
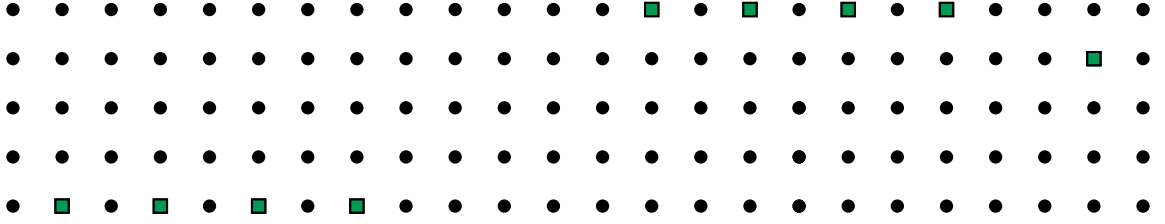


Figure 5.4.8: A  $4 \times 24$  example of  $S_B$ .

For  $S_W$ , consider four cases. If  $m$  is odd and  $n$  is even, we choose the vertices from the white board that correspond to those on the black board when the black board is reflected vertically as seen in Figure 5.4.9. Let:

- $S_W^I = \{(n - 2, m - 1)\}$
- $S_W^D = \{(n - 1 - (3m + 1 + 2i + j(6m - 6)), 0) \mid 0 \leq i \leq m - 2, 0 \leq j \leq \lfloor \frac{n-5m+2}{6m-6} \rfloor\}$

$$\bullet S_W^U = \{(n-1 - (4+2i+j(6m-6)), m-1) \mid 0 \leq i \leq m-2, 0 \leq j \leq \lfloor \frac{n-2m-1}{6m-6} \rfloor\}$$

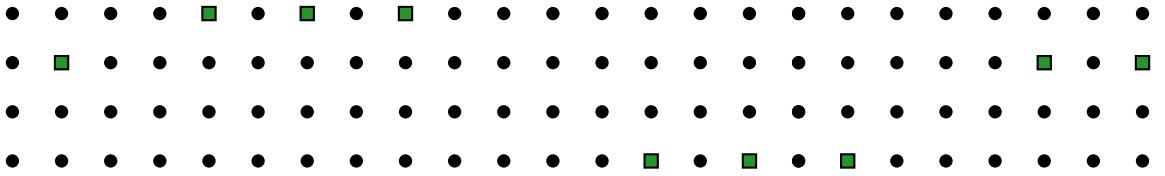
Figure 5.4.9: A  $5 \times 24$  example of  $S_W$ .

If  $m$  is even choose those white vertices that correspond to reflecting the board horizontally as in Figure 5.4.10:

$$\bullet S_W^I = \{(1, m-1)\}$$

$$\bullet S_W^D = \{(3m+1+2i+j^D(6m-6), 0) \mid 0 \leq i \leq m-2, 0 \leq j^D \leq \lfloor \frac{n-2m-1}{6m-6} \rfloor\}$$

$$\bullet S_W^U = \{(4+2i+j(6m-6), m-1), 0 \leq i \leq m-2, 0 \leq j^U \leq \lfloor \frac{n-5m+2}{6m-6} \rfloor\}$$

Figure 5.4.10: A  $4 \times 24$  example of  $S_W$ .

As we can see in Figure 5.4.11, if  $m$  and  $n$  are both odd with  $n = 1 \pmod 4$ , let:

$$\bullet S_W^I = \{(2i+1, \lfloor \frac{m}{2} \rfloor) \mid 0 \leq i \leq \lfloor \frac{m+5}{4} \rfloor\}$$

$$\bullet S_W^D = \{(m+2+2i+j^D(6m-6), 0) \mid 0 \leq i \leq m-2, 0 \leq j^D \leq \lfloor \frac{n-3m+1}{6m-6} \rfloor\}$$

$$\bullet S_W^U = \{(4m-1+2i+j^U(6m-6), m-1) \mid 0 \leq i \leq m-2, 0 \leq j^U \leq \lfloor \frac{n-6m+4}{6m-6} \rfloor\}$$

As seen Figure 5.4.12, if  $m$  and  $n$  are both odd with  $n = 3 \pmod 4$ , let

$$\bullet S_W^I = \{(2i, \lfloor \frac{m}{2} \rfloor) \mid 0 \leq i \leq \lfloor \frac{m+5}{4} \rfloor\}$$

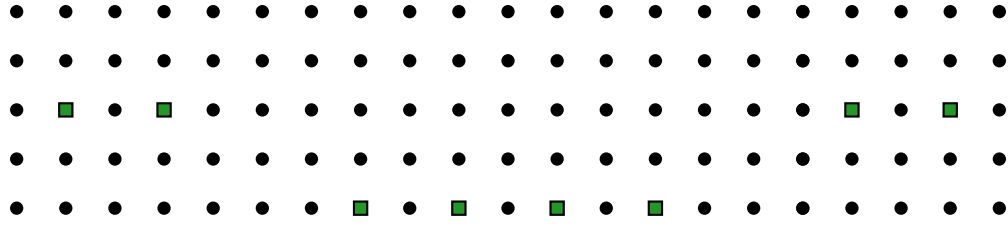


Figure 5.4.11: A  $5 \times 21$  example of  $S_W$ .

- $S_W^D = \{(m + 2 + 2i + j^D(6m - 6), 0) \mid 0 \leq i \leq m - 2, 0 \leq j^D \leq \lfloor \frac{n-3m+1}{6m-6} \rfloor\}$
- $S_W^U = \{(4m - 1 + 2i + j^U(6m - 6), m - 1) \mid 0 \leq i \leq m - 2, 0 \leq j^U \leq \lfloor \frac{n-6m+4}{6m-6} \rfloor\}$

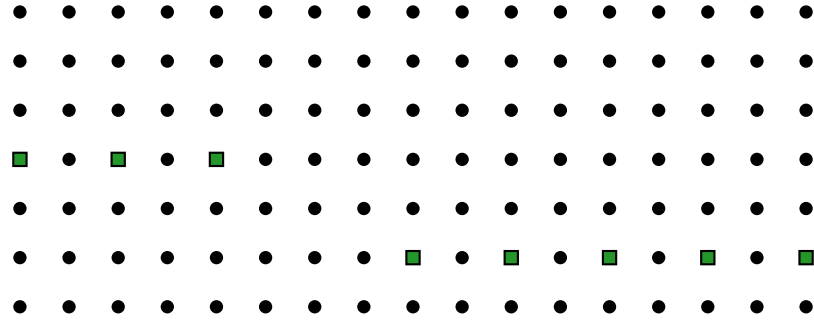


Figure 5.4.12: A  $7 \times 17$  example of  $S_W$ .

In each of the four cases above, we may choose those vertices for  $S_W^T$  similar to those chosen for  $S_B^T$ : the remaining undominated vertices in  $B_W(m, n)$  form a truncated triangle/trapezoid - that is, a triangle or trapezoid with the corner removed. See Figure 5.4.3b for an example.

It remains to be shown that each of these constructions gives the required size as stated in Conjecture 5.4.8 and to show that each of the sets dominates  $B(m, n)$  accordingly.

**Proposition 5.4.9.** [9]  $B_n$  is claw-free.

Since  $m \leq n$  we have  $B(m, n)$  is an induced subgraph of  $B_n$ , which is claw-free by Proposition 5.4.9, and, therefore,  $B(m, n)$  is also claw-free. Also, each of the vertex sets given in Theorem 5.4.3 and Conjecture 5.4.8 can be shown to be independent

sets. Hence  $i(B(m, n)) = \gamma(B(m, n))$ . Thus,  $B(m, n)$  is domination perfect. Based on this conclusion, our upper bound from Theorem 5.4.3, and our constructions we have one final conjecture:

**Conjecture 5.4.10.**

$$\gamma(B(m, n)) = i(B(m, n)) = \begin{cases} 2 \lceil \frac{n-1}{2} \rceil & \text{if } m < n \leq 2m \\ 2 \lfloor \frac{m+n}{3} \rfloor & \text{if } n > 2m \end{cases}$$



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