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Distributive lattice models of the type B elementary Weyl group symmetric functions

Katheryn Beck

Distributive lattice models of the type B elementary Weyl group symmetric functions

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Katheryn Alisande Beck

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ABSTRACT

One of the great themes of algebraic combinatorics is the exploration of connections between ordered structures and group actions/representations. This thesis furthers this theme by presenting diamond-colored distributive lattice models of certain polynomials that are invariant under the action of the type B_n Weyl group. Initially we realize these lattices as diamond-colored lattices of order ideals from certain vertex-colored posets. We explore various coordinatizations of these lattices via partition-like elements, tableaux, and binary-type representations called tally diagrams. We also examine algebraic properties of these lattices. In particular, we prove that our type B_n lattices are effective models for the type B_n elementary Weyl symmetric functions.

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CONTENTS

LIST OF FIGURES

Chapter 1

Introduction

Integer partitions appear in various branches of mathematics, perhaps most conspicuously in enumerative combinatorics and representation theory. In these latter contexts, partial orderings of integer partitions are often fruitfully employed. For one example, Young's lattice is the ordering of all integer partitions by containment of their partition diagrams (also known as Ferrers diagrams). This lattice is a tool for the study of the irreducible representations of the symmetric groups; for R. P. Stanley, Young's lattice was an originating example of objects he named "differential posets," which can be viewed as combinatorial models of Weyl algebras. See [Stan1] for further discussion.

For another example of an effective use of partial orderings of integer partitions, we reprise and amplify in this paper some aspects of the story of a curious family of distributive lattice orderings of certain partitions that are naturally affiliated with the q-binomial coefficients (aka Gaussian polynomials). Lattices in this two-parameter family can be indexed by integers k and N for which $1 \leq k \leq N-1$, where the lattice $L(k, N-k)$ is the set of integer partitions with no more than k parts and with largest part no larger than $N-k$ together with the partial ordering induced by Young's lattice. The rank generating function of $L(k, N-k)$ is easily seen to be the q-binomial coefficient $\binom{N}{k}_q$. Moreover, $L(k, N-k)$ is closely connected to the k^{th} fundamental representation of the simple complex special linear Lie algebra $\mathfrak{sl}(N,\mathbb{C})$, as in [Sag] or [Stan2], as well as the k^{th} elementary symmetric function whose N variables are permuted by the symmetric group S_N, see [Proc2] or [HL]. (Henceforth, we assume Lie algebras are complex.) These connections are effected by a certain natural coloring of the edges of the order diagram (aka Hasse diagram) for $L(k, N-k)$: The colored edges indicate how certain $\mathfrak{sl}(N,\mathbb{C})/S_N$ generators act on basis vectors/monomials associated with vertices of the order diagram.

Now, elementary symmetric functions are special occurrences of Schur functions, and the latter are symmetric functions that can be expressed as "bialternants" (that is, as quotients of certain alternants) under the action of S_N. With reference to the classification of complex simple Lie algebras by Dynkin diagrams (see the so-called finitary GCM graphs of §2 below), we can view S_N as the type A_{N-1} Weyl group and therefore naturally extend to other Weyl groups the general notion of a symmetric function and the more specific notion of a bialternant. In the language of [Don4] and [ADLMPPW], $L(k, N-k)$ is both a "supporting graph" for a fundamental representation of the type A_{N-1} simple Lie algebra and a "splitting poset" for a type A_{N-1} elementary Weyl symmetric function. With these reasons in mind, we shall refer to $L(k, N-k)$ with the aforementioned edge-coloring as a "type A_{N-1} elementary lattice" and use the more specific notation $L_{\mathsf{A}}(k, N-k)$.

That $L_{\mathsf{A}}(k, N-k)$ serves as a combinatorial model for a simple Lie algebra representation and its associated Weyl group symmetric function makes $L_{\mathsf{A}}(k, N-k)$ a distinctive kind of object. This distinction is all the more pronounced given that $L_{\mathsf{A}}(k, N-k)$ is the *only possible* such model for this representation/symmetric function pair, cf. [Don4] and [Don7]. We note that while a supporting graph for a given simple Lie algebra representation is automatically a splitting poset for the associated

Weyl group symmetric function (§4 of [Don7]), the reverse is not always true; see for example [ADLP] and [ADLMPPW]. An ongoing project developed by the advisor of this thesis and many collaborators is to seek out poset models for other simple Lie algebra representations and/or their companion Weyl group symmetric functions. Explicit renderings of such posets are relatively rare and usually quite pretty.

While these algebraic connections provide some of the impetus for the study of the type A elementary lattices, another motivating interest is that they have many desirable combinatorial features which one might hope to generalize. For example, these ranked lattices have rank generating functions expressible as quotients of products; they are rank symmetric, rank unimodal, and strongly Sperner (see e.g. [Proc1]); they are the natural environment for studying a certain move-minimizing game involving dominoes (see [DDS]); and they afford a salutary environment for a combinatorial proof of the unimodality of q-binomial coefficients (see [O] and [Zeil]). Several of the preceding features — namely, rank generating functions expressed as quotients of products; rank symmetry; and rank unimodality — can be viewed as direct consequences of the fact that the type A elementary lattices are splitting posets (actually, splitting distributive lattices) for the type A elementary Weyl symmetric functions. For these reasons, we believe it is beneficial to seek out splitting posets for other families of Weyl symmetric functions, whether or not such posets can serve as models for Lie algebra representations.

Some terminology in the following paragraphs borrows from §2 and §4 below. Associated to each connected finitary GCM graph X_n in Figure 2.1 (where $\mathsf{X} \in \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \mathsf{F}, \mathsf{G}\}$) is a finite irreducible root system, a finite irreducible Weyl group, and a finite-dimensional simple Lie algebra, and for each node γ_k of the graph, there is a corresponding fundamental weight ω_k. The notation $\chi_{\omega_k}^{\mathsf{X}_n}$ refers to the symmetric function associated with the fundamental weight ω_k under the action of the type X_n Weyl group; we call this a "type X_n elementary symmetric function." Some core results about the type A elementary lattices are recapitulated in this paper. Some type C analogs of the type A elementary lattices are provided in [Don2] and [Don3], and from here on, we refer to these as type C elementary lattices. The type C elementary lattices are splitting distributive lattices for the type C elementary Weyl symmetric functions and are supporting graphs for the fundamental representations of the associate type C (symplectic) simple Lie algebra. In this paper we present type B elementary lattices and obtain some key results. Here is a tabular summary:

Finitary GCM graph X_m	Natural/named elementary lattices	Splitting posets for $\chi^{\mathsf{X}_m}_{\omega_k}$?	Supporting graphs for k^{th} fundamental simple Lie algebra representation?
A_{N-1}	$L_{\mathsf{A}}(k, N-k)$ of classical fame	Yes (well-known) (See §2 and 4 below)	Yes (well-known) (See [Proc2], [Don4], [HL], etc)
B_n	$L^{KN}_{\mathsf{B}}(k, 2n+1-k)$ and $L^{DeC}_{\mathsf{B}}(k, 2n+1-k)$ of this paper	Yes (See §2-5 below)	Yes (In preparation by thesis advisor)
C_n	$L^{KN}_{\mathsf{C}}(k, 2n-k)$ and $L^{DeC}_{\mathsf{C}}(k, 2n-k)$ of [Don2] and [Don4]	Yes (Supporting graph $\Rightarrow$ splitting poset)	Yes (See §[Don2] and [Don4])

We know of analogs in other types as well. All minuscule, short adjoint (aka quasi-minuscule), and adjoint simple Lie algebra representations in types B, D, E, F, and G are associated with certain fundamental weights. (The weight associated with a short adjoint (respectively, adjoint) representation is the highest short root (respectively, highest root); in types D and E, all roots have the same length, and the convention is that these are all long.) In each case, splitting modular lattices for the affiliated elementary Weyl symmetric functions are known. See [Don5] for discussion of splitting modular lattices associated with the short adjoint and adjoint representations in all types. See [Don4] for discussion of splitting modular lattices associated with the minuscule representations in all types (E_8, F_4, and G_2 have no minuscule representations); for the minuscule cases, all these modular lattices are, in fact, distributive. That is, for the finitary GCM graph / fundamental weight pairs (X_n, ω_k) in the table below, we have splitting modular/distributive lattices for $\chi_{\omega_k}^{\mathsf{X}_n}$. We believe these should be called elementary lattices.

Finitary GCM graph X_m	D_n	E_6	E_7	E_8	F_4	G_2
Minuscule weights ω_k	$k = 1, n-1, n$	$k = 1, 6$	$k = 7$	—	—	—
Highest short root ω_k	—	—	—	—	$k = 4$	$k = 1$
Highest root ω_k	$k = 2$	$k = 2$	$k = 1$	$k = 8$	$k = 1$	$k = 2$

Note that, besides those cases reported in the above table, type D analogs of the type $\mathsf{A}/\mathsf{B}/\mathsf{C}$ elementary lattices are not known. To us, this is a very intriguing open question, which we formally state as follows:

Open Problem 1.1 Find type D analogs of the type $\mathsf{A}/\mathsf{B}/\mathsf{C}$ elementary lattices.

Before we outline the main content of the paper, we make the following comment on its overall context. While semisimple Lie algebra representation theory is part of the larger context of this work, it is not needed in order to understand the results obtained / methods used here. In this paper, we only employ methodologies of finite poset theory, enumerative combinatorics, general Weyl symmetric function theory, and actions of finite groups.

Within this environment, then, the main goal of this paper is to present type B analogs of the type A elementary lattices and demonstrate that these are splitting distributive lattices for certain type B elementary (and "almost-elementary") Weyl symmetric functions. In particular, in §3, we present two families of edge-colored distributive lattices as partial orderings of certain partitions and/or partition-like objects. These families of lattices are indexed by two integer parameters k and n, with $1 \leq k \leq n$. For fixed k and n, the two type B lattices are denoted $L_{\mathsf{B}}^{KN}(k, 2n+1-k)$ and $L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$. In §3, we extend many of the basic combinatorial properties of $L_{\mathsf{A}}(k, N-k)$ to $L_{\mathsf{B}}^{KN}(k, 2n+1-k)$ and $L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$. The edge coloring presented in §3 will allow us to view weight-generating functions naturally associated with $L_{\mathsf{B}}^{KN}(k, 2n+1-k)$ and $L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$ as certain symmetric functions with respect to the action of the type B_n Weyl group. Our main result (Theorem 5.1) demonstrates, via the application of a vertex-coloring method developed by the Donnelly in [Don7], that $L_{\mathsf{B}}^{KN}(k, 2n+1-k)$ and $L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$ are splitting posets for the k^{th} type B_n elementary Weyl symmetric function, when $k < n$. (When $k = n$, the associated B_n-symmetric function is not elementary but is, nonetheless, a Weyl bialternant.) Definitions and foundational results for some of the crucial combinatorial notions used here are presented in §2; §4 provides some background on the theory of poset models for Weyl group symmetric functions. The type A elementary lattices and a special family of type B lattices will serve to illustrate the concepts of §2 and §4.

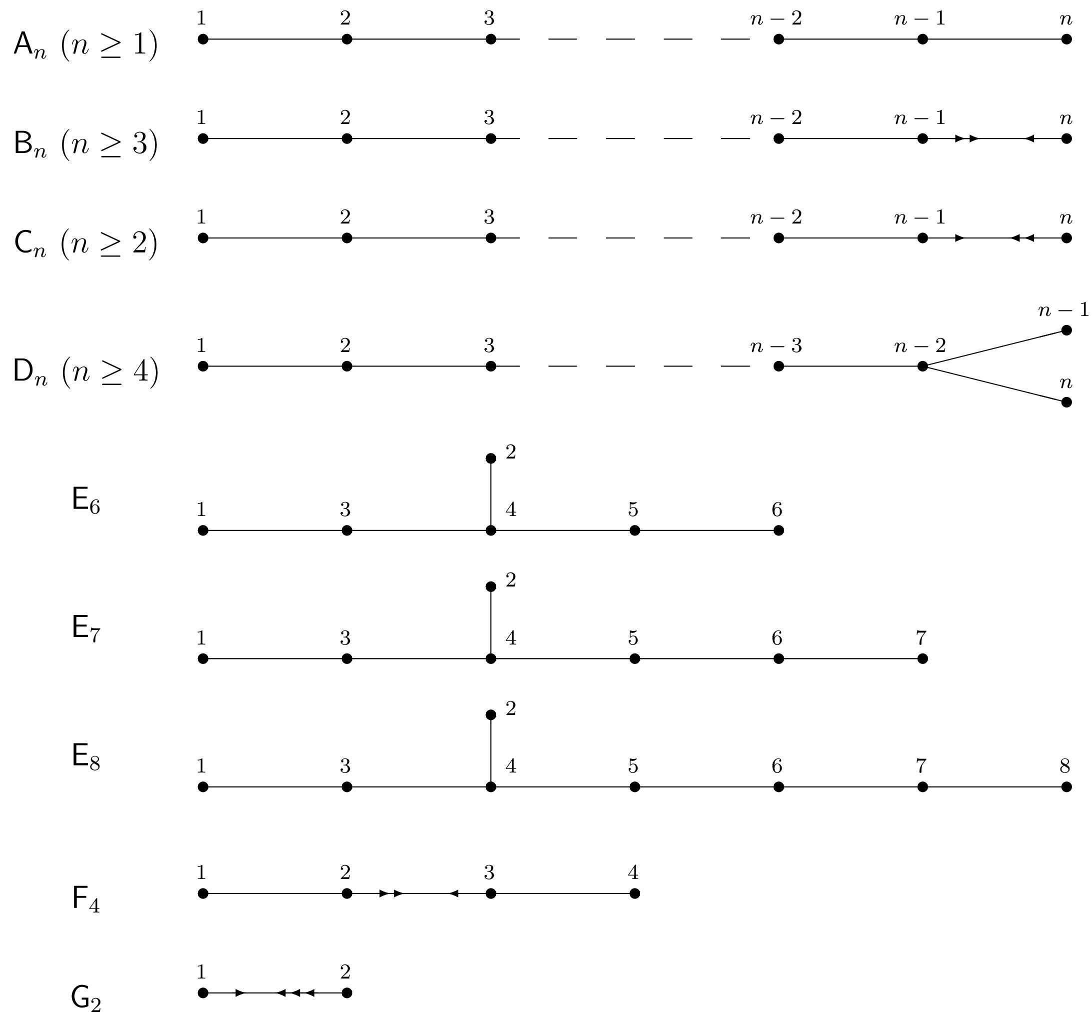
Chapter 2

Combinatorial preliminaries, type A elementary lattices, and a special family of type B lattices

To begin, we make a few comments about the general milieu of our work. Certain finite graph/integer matrix pairs, often referred to as Dynkin diagrams, are identifiers of various classical objects such as root systems, Coxeter groups, and Kac–Moody Lie algebras. For us, a choice of such a diagram, herein referred to as a "GCM graph," will serve to declare the immediate algebraic/combinatorial environment for any results that follow. Thus, as in §1, to say "type A_{N-1}" is to refer to the size $N^2 - N$ root system, the $N!$-element Weyl group (which is the symmetric group S_N), and/or the (N^2-1)-dimensional simple Lie algebra $\mathfrak{sl}(N, \mathbb{C})$ that are each associated with the finitary GCM graph A_{N-1}. For purposes of this chapter, however, the aforementioned algebraic contexts are suppressed and the related purely combinatorial notions are brought to the fore.

Following [Don6], we take as our starting point some given simple graph Γ on n nodes. In particular, Γ has no loops and no multiple edges. Nodes $\{\gamma_i\}_{i \in I_n}$ for Γ are indexed by elements of some fixed totally ordered set I_n of size n (usually $I_n = \{1 < 2 < \cdots < n\}$). For each pair of adjacent nodes γ_i and γ_j in Γ, choose two negative integers a_{ij} and a_{ji}. Extend this to an $n \times n$ matrix $A = (a_{ij})_{i,j \in I_n}$ where,

Figure 2.1: Connected finitary GCM graphs.



in addition to the negative integers a_{ij} and a_{ji} taken from the edges of Γ, we have $a_{ii} := 2$ for all $i \in I_n$ and $a_{ij} := 0$ if there is no edge in Γ between nodes γ_i and γ_j. We call the pair (Γ, A) a *GCM graph*, since A is a 'generalized Cartan matrix' as in [Kac] and [Kum]*.

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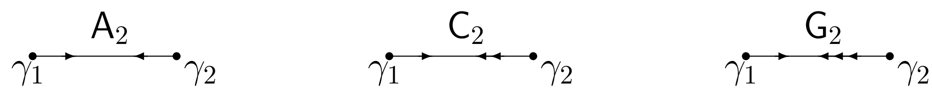
*Such matrices are the starting point for the study of Kac–Moody Lie algebras. For us, these matrices also encode information about root systems and their associated Weyl groups. The latter provide a suitable environment for studying "Weyl symmetric functions," which can be thought of as special multivariate Laurent polynomials which are invariant under a certain natural action of the Weyl group. An overarching goal of our work is to find nice poset models for such Weyl symmetric functions. See §4 for further development of the ideas in the preceding two sentences.

We say a GCM graph (Γ, A) is *connected* if Γ is. We depict a generic connected two-node GCM graph as $\gamma_1 \overset{p}{\longrightarrow}\overset{q}{\longleftarrow} \gamma_2$, where $p = -a_{12}$ and $q = -a_{21}$. Those two-node GCM those graphs which have $p = 1$ and $q = 1$, 2, or 3 (respectively) have special names:



When $p = 1$ and $q = 1$ it is convenient to use the graph $\gamma_1 \frac{\qquad\quad}{} \gamma_2$ to represent the GCM graph A_2. A GCM graph (Γ, A) is *finitary* if each connected component of (Γ, A) (in the obvious sense) is one of the graphs of Figure 2.1. In exactly these cases, the affiliated root system and Weyl group are irreducible and finite and the related Kac–Moody alegbra is simple and finite-dimensional, and we call the matrix A a *Cartan matrix*. We number the nodes of connected finitary GCM graphs as in §11.4 of [Hum]. The special two-node GCM graphs A_2, C_2, and G_2 above are finitary GCM graphs with Cartan matrices $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$, and $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$. We identify any irreducible finite root system (or affiliated irreducible finite Weyl group) by its associated GCM graph name X_n, where $\mathsf{X} \in \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \mathsf{F}, \mathsf{G}\}$.

Poset principles: A précis. Our interest is in finding combinatorially interesting partially ordered sets that exhibit and model aspects of the various algebraic structures related to the connected finitary GCM graphs. So, we should set some notation and terminology relating to posets. Here we provide a brief overview that highlights idiosyncratic or perhaps more specialized poset concepts. We mostly follow the conventions of [DDDS] and references therein, which can be consulted for more detail.

Given a poset P with partial ordering relation "$\leq$" (reflexive, anti-symmetric, transitive), a *covering relation* is an ordered pair of poset elements $(\mathbf{x}, \mathbf{y}) \in P \times P$ with the property that $\mathbf{x} = \mathbf{z}$ or $\mathbf{y} = \mathbf{z}$ whenever $\mathbf{x} \leq \mathbf{z} \leq \mathbf{y}$. We depict the ordered pair $(\mathbf{x}, \mathbf{y})$ as a directed edge $\mathbf{x} \to \mathbf{y}$. The *order diagram* for this poset, also denoted by P, is the directed graph whose vertices are the poset elements and whose directed edges are the covering relations. When needed, we use the notation $\mathcal{V}(P)$ to denote the vertex set of the order diagram and $\mathcal{E}(P)$ to denote the set of directed edges. All posets in this paper are finite. When we depict posets, edges will be directed upward, so arrowheads on directed edges will often not be drawn. We apply graph theoretic notions (connectedness, adjacency of vertices, etc) to a poset by applying them to its order diagram.

A poset R is *ranked* if there is a nonnegative integer ℓ and a surjective function $\rho : R \longrightarrow \{0, 1, \ldots, \ell\}$ for which $\rho(\mathbf{x})+1 = \rho(\mathbf{y})$ for any covering relation $\mathbf{x} \to \mathbf{y}$. The number ℓ is the *length* of R with respect to the *rank function* ρ. The related *depth function* $\delta : R \longrightarrow \{0, 1, \ldots, \ell\}$ is given by $\delta(\mathbf{x}) := \ell - \rho(\mathbf{x})$. (If R is connected, then the rank and depth functions are unique.) This ranked poset is *rank symmetric* if, for each integer $r \in \{0, 1, \ldots, \ell\}$ we have $|\rho^{-1}(r)| = |\rho^{-1}(\ell - r)|$. It is *rank unimodal* if, for some integer $u \in \{0, 1, \ldots, \ell\}$, we have

$$\begin{aligned}|\rho^{-1}(0)| \leq |\rho^{-1}(1)| \leq \cdots \leq |\rho^{-1}(u-1)| \leq |\rho^{-1}(u)| \\ \geq |\rho^{-1}(u+1)| \geq \cdots \geq |\rho^{-1}(\ell-1)| \geq |\rho^{-1}(\ell)|.\end{aligned}$$

We define the *rank generating function* $\mathsf{RGF}(R; q)$ by the rule

$$\mathsf{RGF}(R; q) := \sum_{\mathbf{x} \in R} q^{\rho(\mathbf{x})} = \sum_{i=0}^{\ell} |\rho^{-1}(i)| q^i.$$

A *lattice* L is a poset for which any two given elements $\mathbf{x}$ and $\mathbf{y}$ of L have a (unique) least upper bound, denoted $\mathbf{x} \vee \mathbf{y}$ and called their *join*, and a (unique) greatest lower bound, denoted $\mathbf{x} \wedge \mathbf{y}$ and called their *meet*. Such a lattice is necessarily connected and has a unique maximal element $\mathbf{max}(L)$ and a unique minimal element $\mathbf{min}(L)$. This lattice is *modular* by definition if and only if L is ranked and $\rho(\mathbf{x} \wedge \mathbf{y}) + \rho(\mathbf{x} \vee \mathbf{y}) = \rho(\mathbf{x}) + \rho(\mathbf{y})$ for any $\mathbf{x}, \mathbf{y} \in L$. The lattice L is *distributive* if and only if meets distribute over joins and vice-versa; that is, $\mathbf{x} \wedge (\mathbf{y} \vee \mathbf{z}) = (\mathbf{x} \wedge \mathbf{y}) \vee (\mathbf{x} \wedge \mathbf{z})$ and $\mathbf{x} \vee (\mathbf{y} \wedge \mathbf{z}) = (\mathbf{x} \vee \mathbf{y}) \wedge (\mathbf{x} \vee \mathbf{z})$ for any given $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L$. Any distributive lattice is modular, but not all modular lattices are distributive.

In a distributive lattice L, an element $\mathbf{x}$ is a *join irreducible* if $\mathbf{x}$ covers exactly one other element of L. Let $\mathbf{j}(L)$ be the set of join irreducible elements of L with partial order induced by L; we call $\mathbf{j}(L)$ the *poset of join irreducibles* of L. Now let P be a poset. A subset $\mathcal{X}$ from P is an *order ideal* if, for any $\mathbf{x} \in \mathcal{X}$ and any $\mathbf{x}'$ in P with $\mathbf{x}' \leq \mathbf{x}$, we have $\mathbf{x}' \in \mathcal{X}$. Let $\mathcal{J}(P)$ be the set of order ideals from P partially ordered by subset containment. Since meets and joins in $\mathcal{J}(P)$ are (respectively) just intersections and unions of sets, it is easy to see that $\mathbf{J}(P)$ is a distributive lattice. What is sometimes called The Fundamental Theorem of Finite Distributive Lattices asserts that for any distributive lattice L and any poset P we have:

$$\mathbf{J}\left(\mathbf{j}(L)\right) \textit{ is isomorphic to } L, \textit{ and } \mathbf{j}\left(\mathbf{J}(P)\right) \textit{ is isomorphic to } P.$$

In fact, the preceding notions can be usefully colorized. Let I be a set of order n; for convenience, in the following discussion we take I to be $\{1, 2, \ldots, n\}$. A poset P together with a function $\mathbf{vertexcolor} : \mathcal{V}(P) \longrightarrow I$ is a *vertex-colored poset.* Similarly, P together with a function $\mathbf{edgecolor} : \mathcal{E}(P) \longrightarrow I$ is an *edge-colored poset.* An edge $\mathbf{x} \rightarrow \mathbf{y}$ in P with color $i \in I$ is denoted $\mathbf{x} \xrightarrow{i} \mathbf{y}$. Assuming P is edge-colored and $J \subseteq I$, then the *J-component* of an element $\mathbf{x} \in P$ is the connected subgraph $\mathbf{comp}_J(\mathbf{x})$ of the order diagram of P whose vertices and edges are obtained as follows: The vertices $\mathcal{V}(\mathbf{comp}_J(\mathbf{x}))$ are all those poset elements that can be reached from $\mathbf{x}$ by traversing a path whose edge colors are in J (we disregard edge directions when traversing edges along such a path); the edges $\mathcal{E}(\mathbf{comp}_J(\mathbf{x}))$ are all edges from $\mathcal{E}(P)$ whose colors are in J and which are incident with some vertex in $\mathcal{V}(\mathbf{comp}_J(\mathbf{x}))$. Now suppose R is ranked poset with edges colored by the set I. Then for any $\mathbf{x} \in R$

and any $i \in I$, the i-component $\mathbf{comp}_i(\mathbf{x})$ is ranked with a unique rank function ρ_i and a unique depth function δ_i. We define the weight of $\mathbf{x}$, denoted $wt(\mathbf{x})$, to be the integer n-tuple

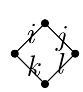
$$wt(\mathbf{x}) = \Big(\rho_i(\mathbf{x}) - \delta_i(\mathbf{x})\Big)_{i \in I}.$$

Let $z_1, z_2, \ldots, z_n$ be variables, and for an integer n-tuple $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ declare that $z^\mu := z_1^{\mu_1} z_2^{\mu_2} \cdots z_n^{\mu_n}$. The *weight generating function* $\mathsf{WGF}(R; z_1, \ldots, z_n)$ in the variables $z_1, \ldots, z_n$ is defined by the rule

$$\mathsf{WGF}(R; z_1, \ldots, z_n) := \sum_{\mathbf{x} \in R} z^{wt(\mathbf{x})}.$$

Now let X_n be a connected finitary GCM graph from Figure 2.1, and for any $i \in I$, let α_i be the i^{th} row vector of the associated Cartan matrix $A = (a_{ij})_{i,j \in I}$. We say the edge-colored ranked poset R is X_n-*structured* if $wt(\mathbf{x}) + \alpha_i = wt(\mathbf{y})$ whenever $\mathbf{x} \xrightarrow{i} \mathbf{y}$ in R. This condition is equivalent to the assertion that for any edge $\mathbf{x} \xrightarrow{i} \mathbf{y}$ in R and for any $j \neq i$, we have

$$\rho_j(\mathbf{x}) - \delta_j(\mathbf{x}) + a_{ij} = \rho_j(\mathbf{y}) - \delta_j(\mathbf{y}).$$

Now suppose L is a distributive lattice whose edges are colored by I. Assign color i to a join irreducible $\mathbf{x}$ if i is the color of the edge beneath $\mathbf{x}$. We denote the resulting vertex-colored poset of join irreducibles by $\mathbf{j}_{color}(L)$. Now, for any poset P that is vertex-colored by I, note that there is a covering relation $\mathcal{X} \rightarrow \mathcal{Y}$ in $\mathbf{J}(P)$ if and only if $\mathcal{X} \subset \mathcal{Y}$ and there exists some $\mathbf{y} \in \mathcal{Y} \setminus \mathcal{X}$ such that $\mathcal{Y} = \mathcal{X} \cup \{\mathbf{y}\}$. If j is the color of the vertex $\mathbf{y}$, then assign color j to the edge $\mathcal{X} \rightarrow \mathcal{Y}$ in the distributive lattice $\mathbf{J}(P)$. We denote the resulting edge-colored distributive lattice by $\mathbf{J}_{color}(P)$. Now, $\mathbf{J}_{color}(P)$ has the property that in any "diamond" of colored edges, parallel edges have the same colors; that is, if  is an edge-colored subgraph of the order diagram of $\mathbf{J}_{color}(P)$, then $i = l$ and $j = k$. We say the distributive lattice $\mathbf{J}_{color}(P)$ is *diamond-colored*. The following is a colorized version of The Fundamental Theorem of Finite Distributive Lattices: Assume that a poset P is vertex-colored by I, that a distributive lattice L is edge-colored by I, and that L is diamond-colored. Then:

$$\mathbf{J}_{color}\left(\mathbf{j}_{color}(L)\right) \textit{ is isomorphic to } L, \textit{ and } \mathbf{j}_{color}\left(\mathbf{J}_{color}(P)\right) \textit{ is isomorphic to } P,$$

where these isomorphisms preserve colors as well as poset structure.

Very often in our work, the combinatorial objects of interest occur naturally as substructures of other objects. In this paragraph, we briefly remark on the poset substructures that are most useful for our purposes here. Given a subset P of a poset Q, let P inherit the partial ordering of Q; call P a *subposet in the induced order*. For posets $(P, \leq_P)$ and $(Q, \leq_Q)$, suppose $P \subseteq Q$ and $\mathbf{x} \leq_P \mathbf{y} \Rightarrow \mathbf{x} \leq_Q \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in P$. Then P is a *weak subposet* of Q. We apply the language of vertex- and/or edge-coloring to subposets in the obvious ways. Now let L be a lattice with partial ordering $\leq_L$ and meet and join operations $\wedge_L$ and $\vee_L$ respectively. Let K be a vertex subset of L. Suppose that K has a lattice partial ordering $\leq_K$ of its own with meet and join operations $\wedge_K$ and $\vee_K$ respectively. We say K is a *sublattice* of L if for all $\mathbf{x}$ and $\mathbf{y}$ in K we have $\mathbf{x} \wedge_K \mathbf{y} = \mathbf{x} \wedge_L \mathbf{y}$ and $\mathbf{x} \vee_K \mathbf{y} = \mathbf{x} \vee_L \mathbf{y}$. It is easy to see that if K is a sublattice of L then for all $\mathbf{x}$ and $\mathbf{y}$ in K we have $\mathbf{x} \leq_K \mathbf{y}$ if and only if $\mathbf{x} \leq_L \mathbf{y}$. That is, K is a weak subposet of L and a subposet in the induced order. If, in addition, K and L are edge-colored and K is an edge-colored weak subposet of L, then call K an *edge-colored sublattice* of L. Whether or not K and L are edge-colored, if K is a sublattice of L, if both K and L are ranked, and if both have the same length, then we say K is a *full-length sublattice* of L. In this case, for any given $\mathbf{x}, \mathbf{y} \in K$, it can be seen that the rank of $\mathbf{x}$ as an element of K is the same as its rank as an element of L and that $\mathbf{y}$ covers $\mathbf{x}$ in K if and only if $\mathbf{y}$ covers $\mathbf{x}$ in L. It can also be seen that if

P is a vertex-colored weak subposet of a vertex-colored poset Q with $|P| = |Q|$, then $\mathbf{J}_{color}(Q)$ is a full-length sublattice of $\mathbf{J}_{color}(P)$.

The type A elementary lattices. Fix integers k and N, with $0 < k < N$. A $k \times (N-k)$ *partition* is an integer k-tuple $\tau = (\tau_1, \tau_2, \ldots, \tau_k)$ for which $N - k \geq \tau_1 \geq \tau_2 \geq \cdots \geq \tau_k \geq 0$. That is, if we view τ as an integer partition (with some parts allowed to be zero), then the partition diagram for τ fits inside a $k \times (N-k)$ grid of boxes. A natural partial ordering of such objects prescribes $\sigma \leq \tau$ for $k \times (N-k)$ partitions $\sigma = (\sigma_1, \ldots, \sigma_k)$ and $\tau = (\tau_1, \ldots, \tau_k)$ if and only if $\sigma_p \leq \tau_p$ for any $p \in \{1, \ldots, k\}$; we say that $\sigma \leq \tau$ *in the componentwise order*. Note that $\sigma \leq \tau$ if and only if the partition diagram for σ is contained in the partition diagram for τ. Let $L(k, N-k)$ be the set of all $k \times (N-k)$ partitions under this partial ordering. It is easy to see that $L(k, N-k)$ is a distributive lattice, with meet and join operations given by $\sigma \wedge \tau = (\min\{\sigma_1, \tau_1\}, \ldots, \min\{\sigma_k, \tau_k\})$ and $\sigma \vee \tau = (\max\{\sigma_1, \tau_1\}, \ldots, \max\{\sigma_k, \tau_k\})$ respectively.

It is crucial to our perspective to view $L(k, N-k)$ as an edge-colored distributive lattice, which we denote $L_{\mathsf{A}}(k, N-k)$ and refer to as the k^{th} *elementary lattice of type* A_{N-1}. Our approach in building this family of edge-colored (in fact diamond-colored) distributive lattices is guided by the principles of §2 of [DDDS] as well as the development of the main example in §3 of that paper. In that spirit, we take as our starting point a certain natural family of vertex-colored posets, from which we build the type A elementary lattices. We will see in Proposition 2.1 that the descriptions of $L_{\mathsf{A}}(k, N-k)$ via partitions (as above) and via order ideals from a vertex-colored poset (as below) are compatible.

Let $P_{\mathsf{A}}(k, N-k)$ denote the poset of pairs $\{(r, c) \,|\, 1 \leq r \leq k, 1 \leq c \leq N-k\}$ partially ordered by component-wise comparison: $(r_1, c_1) \leq (r_2, c_2)$ if and only if

$r_1 \leq r_2$ and $c_1 \leq c_2$. We identify $P := P_{\mathsf{A}}(k, N-k)$ with its order diagram, thought of as a directed graph whose vertex set $\mathcal{V}(P)$ is the pairs (r, c) comprising P and whose set of directed edges $\mathcal{E}(P)$ is the set of covering relations

$$\{(r_1, c_1) \rightarrow (r_2, c_2) \,|\, (r_i, c_i) \in \mathcal{V}(P) \text{ for } i = 1, 2 \text{ and either } r_2 = r_1 + 1 \text{ or } c_2 = c_1 + 1\}.$$

We think of r and c as row-like and column-like indices of the positions of the vertices of P. We color the vertices of P via the function $\mathbf{vertexcolor} : \mathcal{V}(P) \longrightarrow \{1, 2, \ldots, N-1\}$ wherein

$$\mathbf{vertexcolor}(r, c) := r - c + N - k.$$

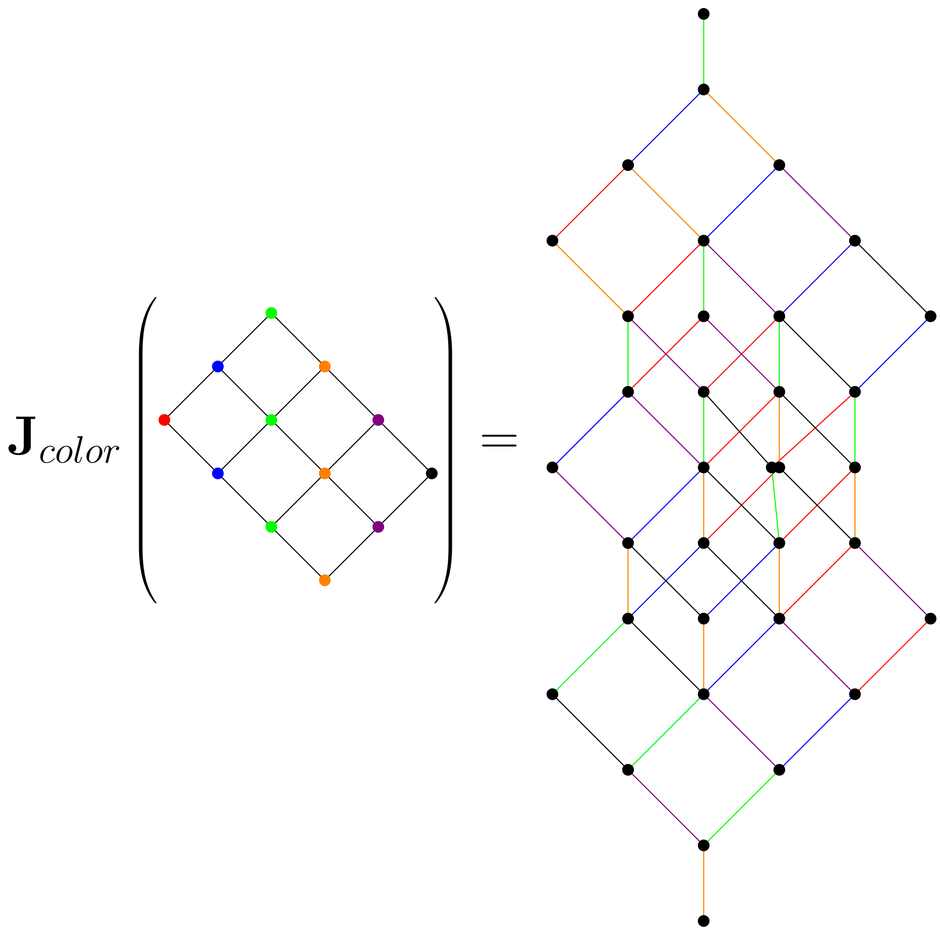
See Figure 2.2 for an example. We can now build $L := L_{\mathsf{A}}(k, N-k)$ from $P = P_{\mathsf{A}}(k, N-k)$ by setting $L := \mathbf{J}_{color}(P)$. That is, L is the diamond-colored distributive lattice of order ideals taken from P.

Next we consider other useful ways to coordinatize the elements of L. In particular, we detail next how to convert order ideals to partitions to columnar tableaux to binary sequences (most often referred to here as "tally diagrams"). These notions are illustrated in Figure 2.3. To effect these conversions, it is convenient at the outset to rotate the order diagram of $P_{\mathsf{A}}(k, N-k)$ clockwise by 135°. From this viewpoint, the element $(r, c) \in P_{\mathsf{A}}(k, N-k)$ occurs at the matrix coordinate location (r, c), and each order ideal $\mathcal{T}$ from $P_{\mathsf{A}}(k, N-k)$ can now easily be coordinatized as a sequence of counts related to the rows of the rotated figure. See Figure 2.3 below.

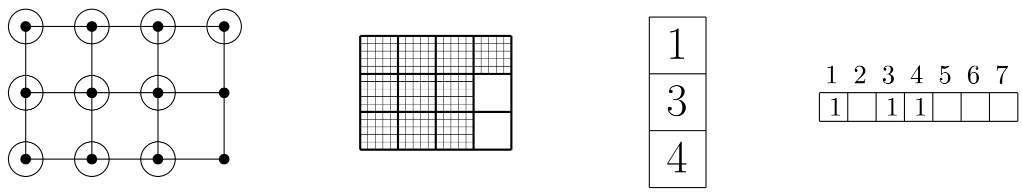
Formally, we associate a partition τ, a column T, and a tally diagram $\mathtt{t}$ to each order ideal $\mathcal{T}$ from $P_{\mathsf{A}}(k, N-k)$ as follows. First, $\tau = \tau(\mathcal{T})$ is the k-tuple given by $\tau_p := |\{(r, c) \in \mathcal{T} : r = p\}|$ when $p \in \{1, 2, \ldots, k\}$. It is evident that τ is a $k \times (N-k)$ partition, and that this process can be reversed to obtain an order ideal from any given $k \times (N-k)$ partition. Second, the *columnar tableau* $T = T(\mathcal{T})$ is

Figure 2.2: Below is the edge-colored lattice $L_{\mathsf{A}}(3,4) = \mathbf{J}_{color}(P_{\mathsf{A}}(3,4))$.

(Vertex and edge colors in this figure are best viewed in a color display.)



the k-tuple for which $T_p = N - k + p - \tau_p$. Note that $1 \leq T_1 < \cdots < T_k \leq N$. Note that this conversion process can be reversed to obtain an $k \times (N-k)$ partition from any given columnar tableau whose k strictly increasing integer entries are from the set $\{1, 2, \ldots, N\}$. The symbol T will simultaneously denote both the column $(T_1, \ldots, T_k)$ and the set $\{T_1, \ldots, T_k\}$. (In general, for k-tuples $X = (X_1, \ldots, X_k)$ and $Y = (Y_1, \ldots, Y_k)$, we say $X \leq Y$ *in the reverse-componentwise order* if and only if $X_p \geq Y_p$ for each $p \in \{1, 2, \ldots, k\}$.) Third, the (type A) *tally diagram* $\mathsf{t} = \mathsf{t}(\mathcal{T})$ associated to the column T is the binary N-tuple $\mathsf{t} = (\mathsf{t}_1, \ldots, \mathsf{t}_N)$, where $\mathsf{t}_p = 1$ if $p \in T$ and $\mathsf{t}_p = 0$ if $p \notin T$, $1 \leq p \leq N$. Of course, $\sum_{i=1}^{N} \mathsf{t}_i = k$. This conversion process can be reversed to obtain a columnar tableau that is a size k subset of $\{1, 2, \ldots, N\}$ from any given binary sequence with N entries that sum to k. We can picture a tally diagram t as follows. Construct a $1 \times N$ grid and index the squares from 1 to N, left to right. Place a unital mark (i.e. a "1") in each of the squares corresponding to elements of the columnar tableau T; all other squares are taken to have a zero and are regarded as empty.



$\mathcal{T}$ $\qquad$ $\tau = (4, 3, 3)$ $\qquad$ T $\qquad$ $\mathsf{t} = (1, 0, 1, 1, 0, 0, 0)$

Figure 2.3: An element of $L_{\mathsf{A}}(3, 4)$.

Tally diagrams (neé "circle diagrams") are an innovation of Sheats [She] that are particularly well-suited to our purposes; these will be the primary objects we use to reference vertices of the lattices considered in this paper. The symbols r, s, t, u etc refer to tally diagrams. For a tally diagram s, the notations $T(\mathsf{s})$, $\tau(\mathsf{s})$, and $\mathcal{T}(\mathsf{s})$ will henceforth have the obvious meanings. The partial order on ideals from $P_{\mathsf{A}}(k, N-k)$ can be realized using partitions, columnar tableaux, and tally diagrams as in the next proposition. The proof is omitted as the details are routine.

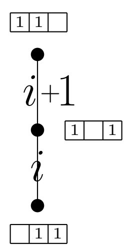
Proposition 2.1 *Let* $\mathsf{s}, \mathsf{t} \in L_{\mathsf{A}}(k, N-k)$ *be tally diagrams, and let* $i \in \{1, 2, \ldots, N-1\}$. *Then*

$$\begin{aligned} \mathcal{T}(\mathsf{s}) \subseteq \mathcal{T}(\mathsf{t}) \quad &\Longleftrightarrow \quad \tau(\mathsf{s}) \leq \tau(\mathsf{t}) \text{ in the componentwise} \\ &\Longleftrightarrow \quad T(\mathsf{s}) \geq T(\mathsf{t}) \text{ in the reverse-componentwise order on } k\text{-tuples} \\ &\Longleftrightarrow \quad \sum_{i=1}^{p} \mathsf{s}_i \leq \sum_{i=1}^{p} \mathsf{t}_i \text{ for all } p \text{ such that } 1 \leq p \leq N. \end{aligned}$$

$$\begin{aligned}
\mathcal{T}(\mathsf{s}) \xrightarrow{i} \mathcal{T}(\mathsf{t}) \quad &\Longleftrightarrow \quad \textit{for some } q \in \{1,2,\ldots,k\},\ \textit{we have } i = q - \tau(\mathsf{t})_q + N - k \\
&\qquad\quad \textit{and } \tau(\mathsf{s})_q + 1 = \tau(\mathsf{t})_q,\ \textit{while } \tau(\mathsf{s})_p = \tau(\mathsf{t})_p \textit{ for } p \neq q \\
&\Longleftrightarrow \quad \textit{for some } q \in \{1,2,\ldots,k\},\ \textit{we have } i = T(\mathsf{t})_q = T(\mathsf{s})_q - 1, \\
&\qquad\quad \textit{while } T(\mathsf{s})_p = T(\mathsf{t})_p \textit{ for } p \neq q \\
&\Longleftrightarrow \quad \mathsf{s}_{i+1} = \mathsf{t}_i = 1, \mathsf{s}_i = \mathsf{t}_{i+1} = 0, \textit{and } \mathsf{s}_p = \mathsf{t}_p \textit{ for } p \notin \{i, i+1\}.
\end{aligned}$$

□

When depicting type A tally diagrams for s and t along an edge $\mathsf{s} \xrightarrow{i} \mathsf{t}$, we will often only depict the portion of each tally diagram corresponding to positions i and $i+1$; we refer to these tally diagram positions together as the i-slots of s or t. Similarly, the $\{i, i+1\}$-slots of a tally diagram consist only of its positions i, $i+1$, and $i+2$. With these conventions in mind, below is a depiction of an $\{i, i+1\}$-component of $L_{\mathsf{A}}(k, N-k)$:

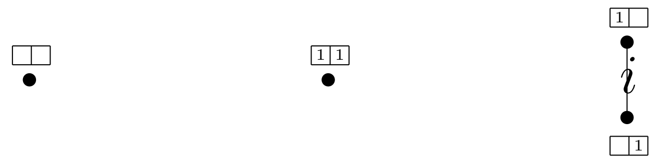


As stated in §1, a principal aim of our work is to connect the diamond-colored distributive lattices studied here to certain symmetric functions associated with the Weyl groups of types A and B. This connection is effected by studying interactions of the edge colors in the lattices, as in the next proposition. While this proposition is purely combinatorial in its statements and proofs, its main significance for our purposes is that it affords an explicit connection to certain type A Weyl symmetric functions, as we will see in §4; in §3, we reformulate the following result as theorem about our proposed type B elementary lattices.

Proposition 2.2 *The following are facts about $L_{\mathsf{A}}(k, N-k)$:*

(1) Let $i \in \{1, 2, \ldots, N-1\}$, thought of as an edge color in $L_{\mathsf{A}}(k, N-k)$. In the

depictions of i-components immediately below, we only depict the i-slots of the tally diagram at any given vertex. Then any i-component of $L_{\mathsf{A}}(k, N-k)$ is one of:



(2) Now let $\mathsf{t} = (\mathsf{t}_1, \ldots, \mathsf{t}_N)$ be a tally diagram in $L_{\mathsf{A}}(k, N-k)$. Then

$$wt(\mathsf{t}) = (\mathsf{t}_1 - \mathsf{t}_2, \mathsf{t}_2 - \mathsf{t}_3, \ldots, \mathsf{t}_{N-1} - \mathsf{t}_N).$$

(3) $L_{\mathsf{A}}(k, N-k)$ is A_{N-1}-structured.

(4) We have the following well-known identity:

$$\mathsf{RGF}(L_{\mathsf{A}}(k, N-k); q) = \begin{pmatrix} N \\ k \end{pmatrix}_q .$$

That is, the rank generating function of $L_{\mathsf{A}}(k, N-k)$ is the q-binomial coefficient $\binom{N}{k}_q$.

Proof. For part *(1)*, note that each of the four possible i-slots can appear in one and only one of the i-components depicted above. Now we prove part *(2)*. Since the i^{th} component of the weight vector $wt(\mathsf{t})$ affiliated with a tally diagram t is completely determined by the combinatorics of the i-component containing t, then it suffices simply to verify that $\rho_i(\mathsf{t}) - \delta_i(\mathsf{t}) = \mathsf{t}_i - \mathsf{t}_{i+1}$ for each of the i-components depicted in part *(1)* of the proposition statement. This is a trivial exercise. To prove *(3)*, we note that it is enough to check that for any distinct edge colors i and j and any edge $\mathsf{s} \xrightarrow{i} \mathsf{t}$, we have

$$\rho_j(\mathsf{s}) - \delta_j(\mathsf{s}) + a_{ij} = \rho_j(\mathsf{t}) - \delta_j(\mathsf{t}), \tag{1}$$

where a_{ij} is the (i, j)-entry of the A_{N-1} GCM with rows/columns indexed in concert with the node labels of the (finitary) GCM graph A_{N-1} from Figure 2.1. Now, by

(2), $\rho_j(\mathsf{x}) - \delta_j(\mathsf{x}) = \mathsf{x}_j - \mathsf{x}_{j+1}$ for any tally diagram x. So, first suppose that i and j are adjacent nodes in the GCM graph A_{N-1} with $j = i+1$, so $a_{i,j} = -1 = a_{j,i}$. Then

$$\mathsf{s} = \boxed{\mathsf{s}_i \mid \mathsf{s}_{i+1} \mid \mathsf{s}_{i+2}} \xrightarrow{i} \boxed{\mathsf{s}_i + 1 \mid \mathsf{s}_{i+1} - 1 \mid \mathsf{s}_{i+2}} = \mathsf{t}.$$

Now, $\rho_{i+1}(\mathsf{s}) - \delta_{i+1}(\mathsf{s}) = \mathsf{s}_{i+1} - \mathsf{s}_{i+2}$, while $\rho_{i+1}(\mathsf{t}) - \delta_{i+1}(\mathsf{t}) = (\mathsf{s}_{i+1} - 1) - \mathsf{s}_{i+2}$. So, $\rho_{i+1}(\mathsf{s}) - \delta_{i+1}(\mathsf{s}) + a_{i,i+1} = \rho_{i+1}(\mathsf{t}) - \delta_{i+1}(\mathsf{t})$, thereby verifying equation (1) above when $j = i+1$. A similar argument establishes (1) under the supposition $j = i-1$. Finally, suppose i and j are distant nodes in the Dynkin diagram for A_{N-1}, so $|j - i| > 1$. Now, t is formed from s by changing entries in the i-slots of s. Since the j-slots of s are distant from the i-slots, then t will have the same entries in its j-slots as s. That is, $\rho_j(\mathsf{s}) - \delta_j(\mathsf{s}) = \rho_j(\mathsf{t}) - \delta_j(\mathsf{t})$. Since $a_{i,j} = 0$ in this case, we get $\rho_j(\mathsf{s}) - \delta_j(\mathsf{s}) + a_{i,j} = \rho_j(\mathsf{t}) - \delta_j(\mathsf{t})$. The identity in *(4)* is well-known; see for example [Proc2]. A standard (and rather pleasant) argument is to use $k \times (N-k)$ partitions to show

$$\mathsf{RGF}(L_{\mathsf{A}}(k, N-k); q) = q^k \mathsf{RGF}(L_{\mathsf{A}}(k, N-1-k); q) + \mathsf{RGF}(L_{\mathsf{A}}(k-1, N-k); q),$$

which coincides with the usual recurrence of q-binomial coefficients:

$$\binom{N}{k}_q = q^k \binom{N-1}{k}_q + \binom{N-1}{k-1}_q.$$

$\square$

A special family of type B lattices. We briefly present here a type B family of lattices that are elsewhere called "minuscule lattices" (see [DDDS] and [Proc2]). Here, we call them "type B-spin elementary lattices" because of their affiliation with certain spin representations of the odd orthogonal Lie algebras/groups, see [KN].

For the remainder of this chapter, n is an integer with $n \geq 2$. As in [DDDS], our starting point is a staircase poset of the kind exhibited in Figure 2.4. Our formal definition utilizes coordinates. Let the set $P_{\mathsf{B}}^{\text{spin}}(n)$ consist of the integer pairs $\{(x, y) \,|\, 1 \leq$

$y \le x \le n$ for integers x and $y\}$. Impose the partial order $(x_1, y_1) \le (x_2, y_2)$ if and only if $x_1 - y_1 \le x_2 - y_2$ with $y_2 \le y_1$. Let **vertexcolor** : $P_{\mathsf{B}}^{\mathrm{spin}}(n) \longrightarrow \{1, 2, \ldots, n\}$ be given by $\mathbf{vertexcolor}(x, y) := x$. (See Figure 2.4 for an illustration of these quantities.) Call the vertex-colored poset $P_{\mathsf{B}}^{\mathrm{spin}}(n)$ the n^{th} *type* B*-spin elementary poset.* Let the diamond-colored distributive lattice $L_{\mathsf{B}}^{\mathrm{spin}}(n) := \mathbf{J}_{color}(P_{\mathsf{B}}^{\mathrm{spin}}(n))$ be called the n^{th} *type* B*-spin elementary lattice.* See Figure 2.5 for a depiction of $L_{\mathsf{B}}^{\mathrm{spin}}(5)$ with two types of coordinates assigned to its vertices. For a lively and informative discussion of the role this family of diamond-colored distributive lattices played in solving an Erdős problem, see R. A. Proctor's article [Proc1], where the n^{th} such lattice is denoted $\mathcal{M}(n)$. For a discussion of these lattices as Lie theoretic objects, see [Proc2], [Don4].

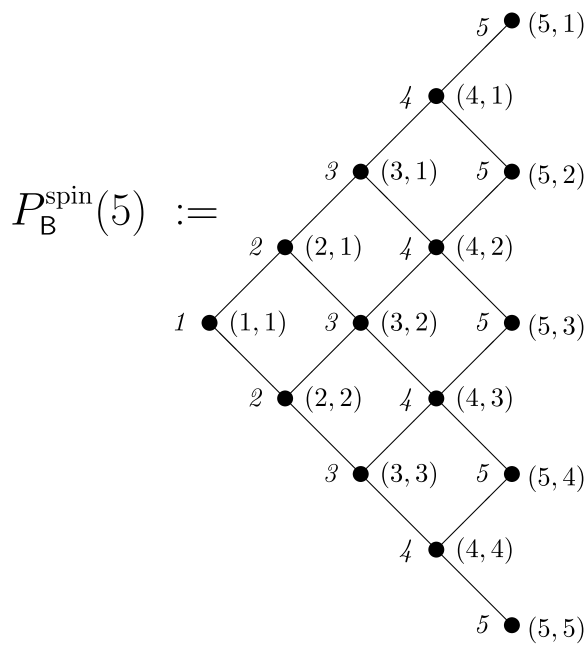


Figure 2.4: The vertex-colored poset $P_{\mathsf{B}}^{\mathrm{spin}}(n)$ with $n = 5$.

(The number to the left of each vertex is its color; to the right are the vertex coordinates.)

It will be advantageous to work with certain coordinatizations of the order ideals from $P_{\mathsf{B}}^{\mathrm{spin}}(n)$. To do so, we designate the n different "southeast to northwest" diagonals of $P_{\mathsf{B}}^{\mathrm{spin}}(n)$ as the elements

$$\mathcal{D}_i^{(\mathsf{B})}(n) := \{(x,y) \in P_{\mathsf{B}}^{\mathrm{spin}}(n) \,|\, i = x - y + 1\},$$

where $1 \leq i \leq n$. Also, we denote by $[1,n]$ the integer set $\{1,2,\ldots,n\}$. Given an order ideal $\mathcal{T}$ from $P_{\mathsf{B}}^{\mathrm{spin}}(n)$, we obtain an n-tuple of integers $\boldsymbol{T}(\mathcal{T}) = \boldsymbol{T} = (\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \ldots, \boldsymbol{\tau}_n)$ wherein $\boldsymbol{\tau}_i = \left|\mathcal{T} \cap \mathcal{D}_i^{(\mathsf{B})}(n)\right|$ for each $i \in [1,n]$. Observe that $n \geq \boldsymbol{\tau}_1 \geq \cdots \geq \boldsymbol{\tau}_n \geq 0$ with $\boldsymbol{\tau}_i > \boldsymbol{\tau}_{i+1}$ if $1 \leq i < n$ and $\boldsymbol{\tau}_i \neq 0$; such an n-tuple is a *zero-cushioned* $[1,n]$*-subset*. Moreover, this conversion process reverses, so that any zero-cushioned $[1,n]$-subset returns an order ideal from $P_{\mathsf{B}}^{\mathrm{spin}}(n)$. Next, we create a binary sequence $\mathsf{t}(\boldsymbol{T}) = \mathsf{t} = (\mathsf{t}_1, \ldots, \mathsf{t}_n)$ from a zero-cushioned $[1,n]$-subset by the rule that t_i is 1 if $n+1-i$ is a member of $\boldsymbol{T}$ and is 0 otherwise. The conversion process clearly reverses, so that any binary n-tuple returns a zero-cushioned $[1,n]$-subset. See Figure 2.5 for a depiction of these coordinates in the $n = 5$ case.

Within this context, we refer to binary n-tuples as *type* B_n*-spin tally diagrams*, and we apply terminology for type A_{N-1} tally diagrams in the obvious ways; one peculiarity of type B-spin tally diagrams is that, in addition to the i-slots when $1 \leq i < n$, we have an n-slot consisting only of the rightmost square/entry. We prefer to think of type B_n-spin tally diagrams as the primary descriptor of the elements in $L_{\mathsf{B}}^{\mathrm{spin}}(n)$. For such a tally diagram x, $\boldsymbol{T}(\mathsf{x})$ is the associated zero-cushioned $[1,n]$-subset and $\mathcal{T}(\mathsf{x})$ is the corresponding order ideal from $P_{\mathsf{B}}^{\mathrm{spin}}(n)$. The next proposition says how to compare elements of $L_{\mathsf{B}}^{\mathrm{spin}}(n)$ via zero-cushioned $[1,n]$-subsets or type B-spin tally diagrams. The proof is omitted, as the supporting details are routine.

Proposition 2.3 *Let* $\mathsf{s}, \mathsf{t} \in L_{\mathsf{B}}^{\mathrm{spin}}(n)$ *be type* B_n*-spin tally diagrams, and let* $i \in \{1,2,\ldots,n\}$*. Then*

$$\mathcal{T}(\mathsf{s}) \subseteq \mathcal{T}(\mathsf{t}) \quad \Longleftrightarrow \quad \boldsymbol{T}(\mathsf{s}) \leq \boldsymbol{T}(\mathsf{t}) \text{ in the componenwise order}$$

$$\iff \quad \sum_{i=1}^{p} \mathsf{s}_i \leq \sum_{i=1}^{p} \mathsf{t}_i \textit{ for all } p \textit{ such that } 1 \leq p \leq n.$$

$$\begin{aligned}
\mathcal{T}(\mathsf{s}) \xrightarrow{i} \mathcal{T}(\mathsf{t}) \quad &\iff \quad \textit{for some } q \in \{1,2,\ldots,n\}, \textit{ we have} \\
&\qquad \textit{and } i = n - \boldsymbol{T}(\mathsf{s})_q, \textit{ while } \boldsymbol{T}(\mathsf{s})_p = \boldsymbol{T}(\mathsf{t})_p \textit{ for } p \neq q \\
&\iff \quad \mathsf{s}_{i+1} = \mathsf{t}_i = 1,\ \mathsf{s}_i = \mathsf{t}_{i+1} = 0, \textit{ and } \mathsf{s}_p = \mathsf{t}_p \textit{ for } p \notin \{i, i+1\}, \\
&\qquad \textit{when } i < n; \textit{ and } \mathsf{s}_n = 0,\ \mathsf{t}_n = 1, \textit{ and } \mathsf{s}_p = \mathsf{t}_p \textit{ for } p \neq n, \\
&\qquad \textit{when } i = n.
\end{aligned}$$

□

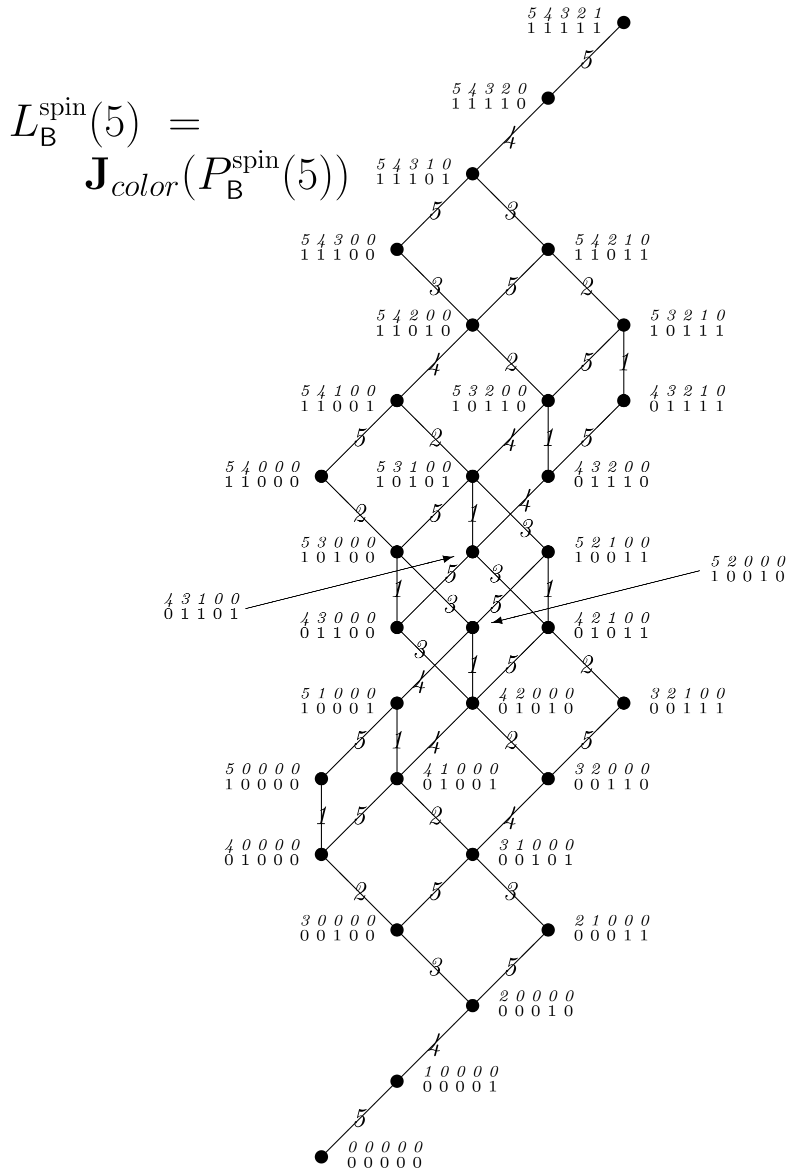


Figure 2.5: The diamond-colored distributive lattice $L_{\mathsf{B}}^{\text{spin}}(n) = \mathbf{J}_{color}(P_{\mathsf{B}}^{\text{spin}}(5))$ with $n = 5$.

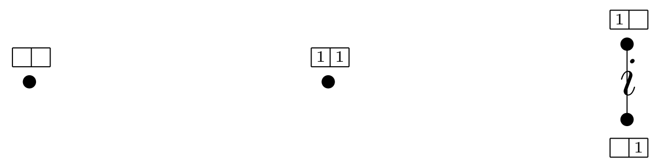
(At each vertex, the *italicized* coordinates are for the associated the zero-cushioned $[1, 5]$-subset; underneath the zero-cushioned $[1, 5]$-subset is the type B-spin tally diagram.)

The next result is an analog of Proposition 2.2.

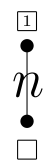
Proposition 2.4 *The following are facts about* $L_{\mathsf{B}}^{\text{spin}}(n)$:

(1) Let $i \in \{1, 2, \ldots, n-1\}$, *thought of as an edge color in* $L_{\mathsf{B}}^{\text{spin}}(n)$. *In the depictions*

of i-components immediately below, we only depict the i-slots of the type B_n-spin tally diagram at any given vertex. Then any i-component of $L_{\mathsf{B}}^{\text{spin}}(n)$ is one of:



Similarly, any n-component of $L_{\mathsf{B}}^{\text{spin}}(n)$ must be:



Above, the only slots depicted are the n-slots of the two type B_n-spin tally diagrams, since all their other slots are identical.

(2) Now let $\mathsf{t} = (\mathsf{t}_1, \ldots, \mathsf{t}_n)$ be a type B_n-spin tally diagram in $L_{\mathsf{B}}^{\text{spin}}(n)$. Then

$$wt(\mathsf{t}) = (\mathsf{t}_1 - \mathsf{t}_2, \mathsf{t}_2 - \mathsf{t}_3, \ldots, \mathsf{t}_{n-1} - \mathsf{t}_n, \mathsf{t}_n).$$

(3) $L_{\mathsf{B}}^{\text{spin}}(n)$ is B_n-structured.

(4) We have the following well-known identity:

$$\mathsf{RGF}(L_{\mathsf{B}}^{\text{spin}}(n); q) = \prod_{i=1}^{n} (1 + q^i).$$

Proof. For part *(1)*, each of the four possible i-slots can appear in one and only one of the i-components depicted above, when $1 \leq i < n$. When $i = n$, there are only two possible n-slots ($\square$ and $\boxed{1}$), and these must appear together in the n-component depicted above. Now we prove part *(2)*. Since the i^{th} coordinate of $wt(\mathsf{t})$ is completely determined by the combinatorics of the i-component containing t, then it suffices to verify that $\rho_i(\mathsf{t}) - \delta_i(\mathsf{t})$ is the appropriate right-hand-side quantity from the identity claimed in *(2)*. When $1 \leq i < n$, one can verify that $\rho_i(\mathsf{t}) - \delta_i(\mathsf{t}) = \mathsf{t}_i - \mathsf{t}_{i+1}$ by checking all of the cases depicted in part *(1)*. Similarly verify that $\rho_n(\mathsf{t}) - \delta_n(\mathsf{t}) = \mathsf{t}_n$ when $i = n$.

To prove *(3)*, we note that it is enough to check that for any distinct edge colors i and j and any edge $\mathsf{s} \xrightarrow{i} \mathsf{t}$, we have

$$\rho_j(\mathsf{s}) - \delta_j(\mathsf{s}) + a_{ij} = \rho_j(\mathsf{t}) - \delta_j(\mathsf{t}),$$

where a_{ij} is the (i,j)-entry of the B_n Cartan matrix with rows/columns indexed in concert with the node labels of the (finitary) GCM graph B_n from Figure 2.1. To establish the preceding identity, we consider certain components of $L_{\mathsf{B}}^{\text{spin}}(n)$. Let $I = \{1, 2, \ldots, n\}$, let $J := I \setminus \{1\}$, and let $J' := I \setminus \{n\}$. Also, for any integer $m \geq 2$, let $\mathbf{max}_m$ (respectively, $\mathbf{min}_m$) denote the unique maximal (respectively, minimal) element of $L_{\mathsf{B}}^{\text{spin}}(m)$. It follows from the description of covering relations in Proposition 2.3 that $\mathbf{comp}_J(\mathbf{max}_n) \cong L_{\mathsf{B}}^{\text{spin}}(n-1)$ and $\mathbf{comp}_J(\mathbf{min}_n) \cong L_{\mathsf{B}}^{\text{spin}}(n-1)$, when $n \geq 3$. An easy induction argument on n allows us to conclude that $\rho_j(\mathsf{s}) - \delta_j(\mathsf{s}) + a_{ij} = \rho_j(\mathsf{t}) - \delta_j(\mathsf{t})$ for all $i, j \in J$. Next, for each $k \in J'$, let $\mathbf{k} := (1, \ldots, 1, 0, \ldots, 0)$, a binary n-tuple with exactly k 1's, all of which are consecutive at the beginning of the sequence. It follows from the descriptions of covering relations in Proposition 2.1 and Proposition 2.3 that $\mathbf{comp}_{J'}(\mathbf{k}) \cong L_{\mathsf{A}}(k, n-k)$. The only other J'-components are the singletons $\mathbf{comp}_{J'}(\mathbf{max}_n) = \{\mathbf{max}_n\}$ and $\mathbf{comp}_{J'}(\mathbf{min}_n) = \{\mathbf{min}_n\}$. It follows that $\rho_j(\mathsf{s}) - \delta_j(\mathsf{s}) + a_{ij} = \rho_j(\mathsf{t}) - \delta_j(\mathsf{t})$ for all $i, j \in J'$. So, it only remains to be checked that $\rho_j(\mathsf{s}) - \delta_j(\mathsf{s}) + a_{ij} = \rho_j(\mathsf{t}) - \delta_j(\mathsf{t})$ for $i, j \in \{1, n\}$. This follows readily from an argument patterned after the proof of part *(3)* of Proposition 2.2.

The expression for the rank generating function in *(4)* is well-known and easily demonstrated using a recurrence related to the decomposition of $L_{\mathsf{B}}^{\text{spin}}(n)$ into J-components. Of course, it is easy to see that $\mathsf{RGF}(L_{\mathsf{B}}^{\text{spin}}(2); q) = 1 + q + q^2 + q^3 = (1 + q)(1 + q^2)$. Now take as an induction hypothesis the claim that $\mathsf{RGF}(L_{\mathsf{B}}^{\text{spin}}(n-1); q) = \prod_{i=1}^{n-1}(1 + q^i)$, where the integer $n \geq 3$. As we observed above $\mathbf{comp}_J(\mathbf{max}_n) \cong L_{\mathsf{B}}^{\text{spin}}(n-1)$ and $\mathbf{comp}_J(\mathbf{min}_n) \cong L_{\mathsf{B}}^{\text{spin}}(n-1)$. Since the rank of $\mathbf{max}_n$ in $L_{\mathsf{B}}^{\text{spin}}(n)$ is

n more than the rank of $\mathbf{max}_{n-1}$ in $L_{\mathsf{B}}^{\text{spin}}(n-1)$, we obtain:

$$\begin{aligned}
\mathsf{RGF}(L_{\mathsf{B}}^{\text{spin}}(n);q) &= \mathsf{RGF}(L_{\mathsf{B}}^{\text{spin}}(n-1);q) + q^n \mathsf{RGF}(L_{\mathsf{B}}^{\text{spin}}(n-1);q) \\
&= \prod_{i=1}^{n-1}(1+q^i) + q^n \prod_{i=1}^{n-1}(1+q^i) \\
&= \prod_{i=1}^{n}(1+q^i). \qquad \square
\end{aligned}$$

Chapter 3

Type B analogs of the type A elementary lattices

Our main objective in this chapter is to produce type B analogs of the type A combinatorial objects/results of §2. To that end, the nomenclature and notation below parallels the conventions of §2. The main result of this chapter is Theorem 3.5, whose part *(4)* recapitulates a bijection that has only appeared in [Don1].

For the remainder of this chapter, k and n are fixed positive integers with $1 \leq k \leq n$. To each such pair of integers we associate three diamond-colored distributive lattices, denoted $\widetilde{L}_{\mathsf{B}}(k, 2n+1-k)$, $L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$, and $L_{\mathsf{B}}^{KN}(k, 2n+1-k)$. These B_n-structured lattices were first discerned by Donnelly, cf. [Don1]. Their direct connection with type B_n Weyl bialternants (independent of their connection to the representation theory of the orthogonal Lie algebras) is explored for the first time here. (Explicit constructions of the fundamental representations of the odd orthogonal Lie algebras on both families of type B elementary lattices are known to Donnelly and will be explored in a future paper.) The lattice $L_{\mathsf{B}}^{KN}(k, 2n+1-k)$ can be built from certain columnar tableaux developed by Kashiwara and Nakashima [KN] in their work on crystal graphs; this lattice has many parallels with the type C lattice $L_{\mathsf{C}}^{KN}(k, 2n-k)$ studied in [Don2] and [Don3]. The type B "De Concini lattice" $L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$ is so named because of its many parallels with the type C lattice $L_{\mathsf{C}}^{DeC}(k, 2n-k)$

of [Don2] and [Don3]; disregarding edge colors, $L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$ is the same as $L_{\mathsf{A}}(k, 2n+1-k)$. The lattice $\widetilde{L}_{\mathsf{B}}(k, 2n+1-k)$ serves as a useful generalization of $L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$ and $L_{\mathsf{B}}^{KN}(k, 2n+1-k)$, as the latter are full-length sublattices of $\widetilde{L}_{\mathsf{B}}(k, 2n+1-k)$.

To begin, we consider vertex-colored posets of join irreducibles that are type B variations of the poset $P_{\mathsf{A}}(k, 2n+1-k)$. To the order ideals from these posets we associate certain partition-like k-tuples, columnar tableaux, and (most usefully) tally diagrams.

Definition 3.1 Let $\widetilde{P}_{\mathsf{B}}(k, 2n+1-k) := P_{\mathsf{A}}(k, 2n+1-k)$ as a set. Write $(r, c) \leq (s, d)$ for two elements of $\widetilde{P}_{\mathsf{B}}(k, 2n+1-k)$ if one of the following holds:

(1) $r = s$ and $c \leq d$

(2) $c = d$, $r \leq s$, and if $r \leq d-n+k-1$, then $s \leq d-n+k-1$.

Define the order on $\widetilde{P}_{\mathsf{B}}(k, 2n+1-k)$ to be the transitive closure of the above relations. It is easy to see that we obtain a partial ordering whose covering relations are as depicted in the first of the three example posets of Figure 3.1. Define a vertex-coloring function by the rule

$$\mathbf{vertexcolor}(r, c) := \begin{cases} r - c + 2n + 1 - k & \text{if } r - c \leq k - n - 1 \\ -(r-c) + k & \text{if } r - c > k - n - 1 \end{cases}$$

It is easy to see that we have $\mathbf{vertexcolor}(r, c) \in \{1, 2, \ldots, n\}$ for any $(r, c) \in \widetilde{P}_{\mathsf{B}}(k, 2n+1-k)$. See Figure 3.1. If we momentarily disregard vertex colors, we see that the poset $\widetilde{P}_{\mathsf{B}}(k, 2n+1-k)$ is a weak subposet of $P_{\mathsf{A}}(k, 2n+1-k)$, since condition (2) just removes some relations from $P_{\mathsf{A}}(k, 2n+1-k)$. Define $\widetilde{L}_{\mathsf{B}}(k, 2n+1-k) := \mathbf{J}_{color}(\widetilde{P}_{\mathsf{B}}(k, 2n+1-k))$. $\boxed{\text{End definition}}$

Next, we define vertex-colored posets $P_{\mathsf{B}}^{KN}(k, 2n+1-k)$ and $P_{\mathsf{B}}^{DeC}(k, 2n+1-k)$ by adding relations to the poset $\widetilde{P}_{\mathsf{B}}(k, 2n+1-k)$ in a manner that is analogous to

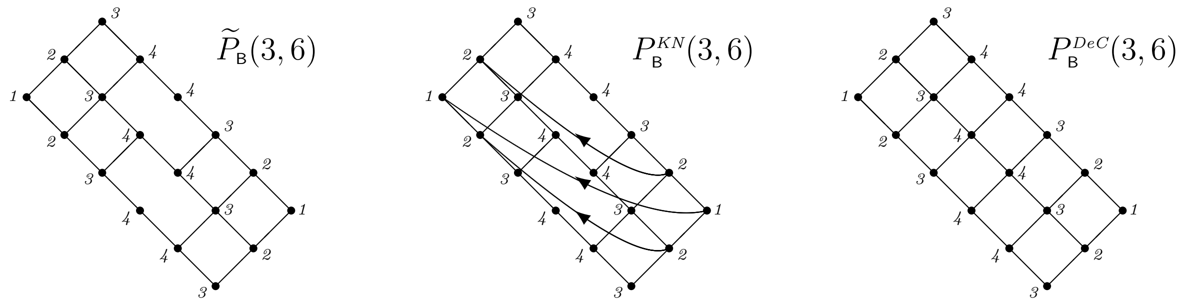


Figure 3.1: Examples of vertex-colored posets of join irreducibles for $n = 4$, $k = 3$

the constructions of the posets $P_{\mathsf{C}}^{KN}(k, 2n-k)$ and $P_{\mathsf{C}}^{DeC}(k, 2n-k)$ from $P_{\mathsf{A}}(k, 2n-k)$ in [Don2].

Definition 3.1.KN Let $P_{\mathsf{B}}^{KN}(k, 2n+1-k) := P(k, 2n+1-k)$ as a set. Write $(r, c) \leq (s, d)$ for two elements of $P_{\mathsf{B}}^{KN}(k, 2n+1-k)$ if one of the following holds:

(1) $(r, c) \leq (s, d)$ in the poset $\widetilde{P}_{\mathsf{B}}(k, 2n+1-k)$

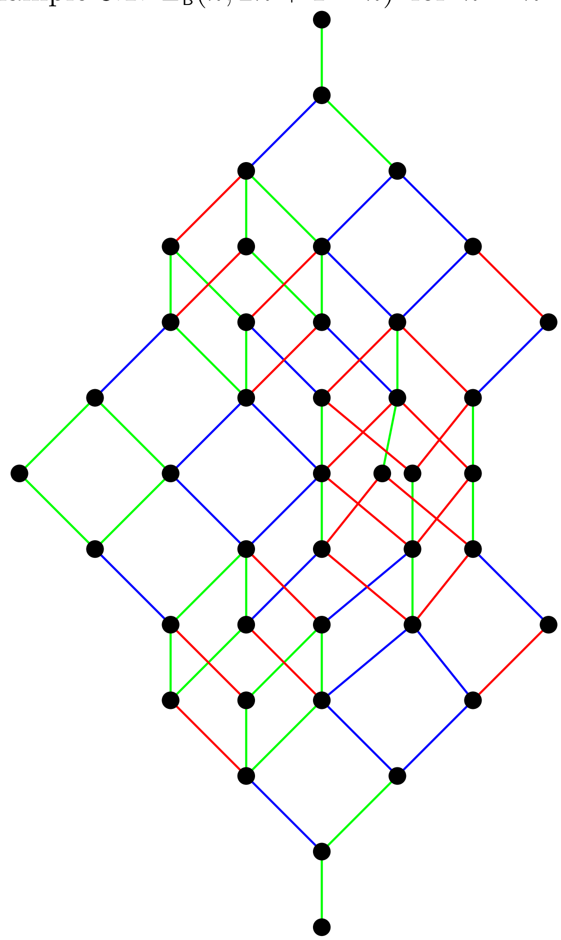
(2) $r > c$ with $(s, d) = (c, 2n+1-2k+r)$.

Define the order on $P_{\mathsf{B}}^{KN}(k, 2n+1-k)$ to be the transitive closure of the above relations. Color the vertices of $P_{\mathsf{B}}^{KN}(k, 2n+1-k)$ as in $\widetilde{P}_{\mathsf{B}}(k, 2n+1-k)$. Define $L_{\mathsf{B}}^{KN}(k, 2n+1-k) := \mathbf{J}_{color}(P_{\mathsf{B}}^{KN}(k, 2n+1-k))$. End definition

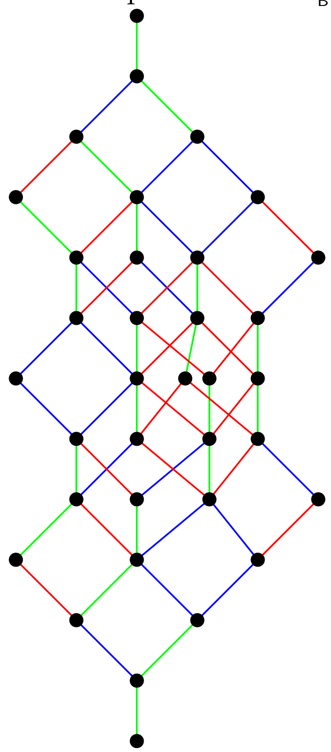
Definition 3.1.DeC Let $P_{\mathsf{B}}^{DeC}(k, 2n+1-k) := P_{\mathsf{A}}(k, 2n+1-k)$ as a poset, but with vertices colored as in $\widetilde{P}_{\mathsf{B}}(k, 2n+1-k)$. Then set $L_{\mathsf{B}}^{DeC}(k, 2n+1-k) := \mathbf{J}_{color}(P_{\mathsf{B}}^{DeC}(k, 2n+1-k))$. End definition

See the following for examples.

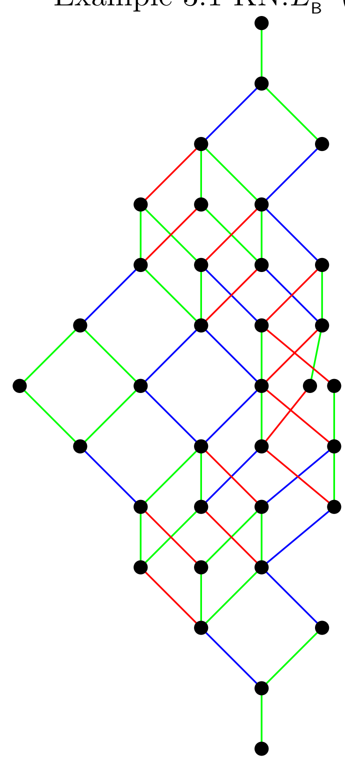
Example 3.1: $\widetilde{L}_{\mathrm{B}}(k, 2n+1-k)$ for $k = n = 3$



Example 3.1 DeC:$L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$ for $k = n = 3$



Example 3.1 KN:$L_{\mathsf{B}}^{KN}(k, 2n+1-k)$ for $k = n = 3$



Next, we describe the partition-like objects, the columnar tableaux, and the tally diagrams associated to the lattices $\widetilde{L}_{\mathsf{B}}(k, 2n+1-k)$, $L_{\mathsf{B}}^{KN}(k, 2n+1-k)$, and $L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$. For $\widetilde{L}_{\mathsf{B}}(k, 2n+1-k)$ and $L_{\mathsf{B}}^{KN}(k, 2n+1-k)$ it often turns out that the k-tuple τ we will associate to an order ideal will not quite be a partition, since $\widetilde{P}_{\mathsf{B}}(k, 2n+1-k)$ was obtained from $P_{\mathsf{A}}(k, 2n+1-k)$ by removing certain covering relations. This latter consideration motivates the following definition.

An integer k-tuple $\tau = (\tau_1, \ldots, \tau_k)$ is a $k \times (2n+1-k)$-*quasi-partition* if the following hold:

(1) $0 \leq \tau_i \leq 2n+1-k$ for $1 \leq i \leq k$.

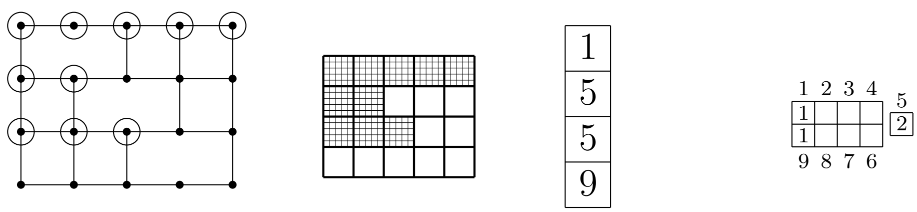
(2) $\tau_i \geq \tau_{i+1}$ unless $\tau_i = n-k+i$ and $\tau_{i+1} = n-k+i+1$ for $1 \leq i < k$.

For a $k \times (2n+1-k)$ quasi-partition τ, let τ' be the $(2n+1-k)$-tuple defined by $\tau'_j := \max\{i : j \leq \tau_i\}$, if this set is non-empty; otherwise, $\tau'_j := 0$. Notice that τ' is a $(2n+1-k) \times k$ *partition*. We say that the $k \times (2n+1-k)$ quasi-partition τ is *Andrews* if $\tau_i - \tau'_i \leq 2n+1-2k$ for $1 \leq i \leq \delta$, where $\delta \times \delta$ is the size of the Durfee square of τ'.

For each order ideal $\mathcal{T}$ taken from $\widetilde{P}_{\mathsf{B}}(k, 2n+1-k)$, define the k-tuple $\tau = \tau(\mathcal{T})$ by $\tau_i := \left|\{(r,c) \in \mathcal{T} : r = i\}\right|$. It can be seen that τ is a $k \times (2n+1-k)$ quasi-partition. Conversely, if τ is a $k \times (2n+1-k)$ quasi-partition, then it can be seen that the vertex set $\mathcal{T} := \{(r,c) : 1 \leq r \leq k, 1 \leq c \leq \tau_r\}$ constitutes an order ideal from $\widetilde{P}_{\mathsf{B}}(k, 2n+1-k)$. For a $k \times (2n+1-k)$ quasi-partition τ, form the complement $\bar{\tau}$ by defining $\bar{\tau}_i := 2n+1-k-\tau_{k+1-i}$. Then $\bar{\tau}$ is a $k \times (2n+1-k)$ quasi-partition. While our use of $\bar{\tau}$ in the sequel is somewhat technical, we note that $\bar{\tau}$ is also quite natural within this lattice setting. For example, it can be shown that $\widetilde{L}_{\mathsf{B}}(k, 2n+1-k)$ is self-dual under the map $\tau \mapsto \bar{\tau}$.

Associate a columnar tableau $T = T(\mathcal{T})$ to the $k \times (2n+1-k)$ quasi-partition τ by "reversing and strictifying" $\bar{\tau}$: That is, T is the integer k-tuple $(T_1, \ldots, T_k)$

wherein $T_i = 2n + 1 - k + i - \tau_i$ for $i \in \{1, \ldots, k\}$. We visualize T as a tableau with k rows, each row containing one box, and with the box in row i (counting from the top) containing the entry T_i. Using the fact that τ is a quasi-partition, it is easy to see that $1 \leq T_1 \leq \cdots \leq T_k \leq 2n + 1$, with $T_i = T_{i+1} \iff T_i = T_{i+1} = n + 1$. This process easily reverses, so that any columnar tableau satisfying the requirements of the preceding sentence yields a $k \times (2n + 1 - k)$ quasi-partition.



$\mathcal{T}$ $\quad \tau = (5, 2, 3, 0)$ $\quad T$ $\quad \mathsf{t} = (1, 0, 0, 0, 2, 0, 0, 0, 1)$

Figure 3.2: A DeC-*in*admissible and KN-*in*admissible element of $\widetilde{L}_{\mathsf{B}}(4, 5)$.

The tally diagram $\mathsf{t} = (\mathsf{t}_1, \ldots, \mathsf{t}_{2n+1})$ associated to the column T is given by $\mathsf{t}_i = 1$ if $i \in T$ and $\mathsf{t}_i = 0$ otherwise, when $i \neq n + 1$. Let $\mathsf{t}_{n+1} = \left|\{i : T_i = n + 1\}\right|$. To picture t, draw a $2 \times n$ grid and attach an additional square just to its right. Index the top row of the $2 \times n$ grid from 1 to n, left to right. Index the bottom row from $2n + 1$ down to $n + 2$, left to right. The rightmost square is indexed $n + 1$. Place tallies in the squares of the tally diagram as usual. The ith slot of t is now the pair $(\mathsf{t}_i, \mathsf{t}_{2n+2-i})$ when $1 \leq i \leq n$ and the singleton t_{n+1} when $i = n + 1$. The other notational conventions of §2 apply straightforwardly to these type B quasi-partitions, columnar tableaux, and tally diagrams.

We say a tally diagram $\mathsf{t} \in \widetilde{L}_{\mathsf{B}}(k, 2n + 1 - k)$ (or a column $T(\mathsf{t})$ or partition $\tau(\mathsf{t})$) is *KN-admissible* (resp. *DeC-admissible*) $\iff$ $\mathcal{T}(\mathsf{t})$ is an order ideal from $P_{\mathsf{B}}^{KN}(k, 2n{+}1{-}k)$ (resp. $P_{\mathsf{B}}^{DeC}(k, 2n{+}1{-}k)$). For example, in Figure 3.2, removing $(3, 3)$ from $\mathcal{T}(\mathsf{t})$ makes t DeC-admissible. Adding $(4, 1)$ to $\mathcal{T}(\mathsf{t})$ makes t KN-admissible.

Analogizing Proposition 3.2 of [Don2], the next proposition explicitly characterizes $L_{\mathsf{B}}^{KN}(k, 2n+1-k)$ and $L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$ as sublattices of $\widetilde{L}_{\mathsf{B}}(k, 2n+1-k)$ in three ways: as sublattices of tally diagrams, of columnar tableaux, and of quasi-partitions.

Proposition 3.2 *Let* t *be a tally diagram in* $\widetilde{L}_{\mathsf{B}}(k, 2n+1-k)$ *with corresponding order ideal* $\mathcal{T}(\mathsf{t})$ *from* $\widetilde{P}_{\mathsf{B}}(k, 2n+1-k)$*. Then*

$\mathcal{T}(\mathsf{t})$ *is an order ideal from* $P_{\mathsf{B}}^{KN}(k, 2n+1-k)$

$\iff$ $\tau(\mathsf{t})$ *is a* $k \times (2n+1-k)$ *Andrews quasi-partition*

$\iff$ $a+k+1-b \leq p$ *whenever* $1 \leq p \leq n$, $T(\mathsf{t})_a = p$*, and* $T(\mathsf{t})_b = 2n+2-p$

$\iff$ $\sum_{p=1}^{q}(\mathsf{t}_p + \mathsf{t}_{2n+2-p}) \leq q,$ *for* $1 \leq q \leq n$.

$\mathcal{T}(\mathsf{t})$ *is an order ideal from* $P_{\mathsf{B}}^{DeC}(k, 2n+1-k)$

$\iff$ $\tau(\mathsf{t})$ *is a* $k \times (2n+1-k)$ *partition*

$\iff$ $T(\mathsf{t})_p < T(\mathsf{t})_{p+1}$ *for* $1 \leq p < k$

$\iff$ $\mathsf{t}_{n+1} \leq 1$.

Proof. The description of DeC-admissibility here is nothing new, since $P_{\mathsf{B}}^{DeC}(k, 2n+1-k)$ is just $P_{\mathsf{A}}(k, 2n+1-k)$ when we ignore vertex colors. The proof of the KN case here is virtually identical to the proof of the KN case of Proposition 3.2 of [Don2]. Follow the same steps of that proof and use the following fact as needed: if $\tau_i < n-k+i$ for a $k \times (2n+1-k)$ quasi-partition τ, then $\tau_j \leq \tau_i$ for all $j \geq i$. $\square$

Replacing $2n+2-i$ with $\bar{i}$ when $1 \leq i \leq n$ and $n+1$ with 0, our total order on column entries becomes $1 < 2 < \cdots < n < 0 < \bar{n} < \cdots < \bar{1}$. Then the KN-admissibility requirement on a columnar tableau $T(\mathsf{t})$ as expressed in Proposition 3.2 becomes exactly the defining condition for the odd orthogonal columns presented by Kashiwara and Nakashima in [KN].

Remark 3.3 Proposition 3.2 says that a tally diagram $\mathsf{t} \in \widetilde{L}_{\mathsf{B}}(k, 2n+1-k)$ is KN-admissible $\Longleftrightarrow$ the total tally in the first q slots is no more than q $\Longleftrightarrow$ there are no more full slots than empty slots among the first q slots, where $1 \leq q \leq n$. □

Proposition 3.4 *Let L be one of $\widetilde{L}_{\mathsf{B}}(k, 2n+1-k)$, $L_{\mathsf{B}}^{KN}(k, 2n+1-k)$, or $L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$. Let s and t be tally diagrams in L. Then:*

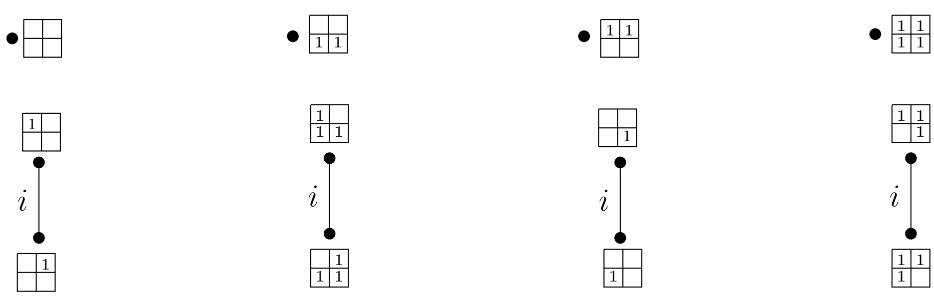
$$\begin{aligned}
\mathcal{T}(\mathsf{s}) \subseteq \mathcal{T}(\mathsf{t}) \quad &\Longleftrightarrow \quad \tau(\mathsf{s}) \leq \tau(\mathsf{t}) \text{ in the componentwise order on } k\text{-tuples} \\
&\Longleftrightarrow \quad T(\mathsf{s}) \geq T(\mathsf{t}) \text{ with } k\text{-tuples ordered reverse-componentwise} \\
&\Longleftrightarrow \quad \sum_{p=1}^{q} \mathsf{s}_p \leq \sum_{p=1}^{q} \mathsf{t}_p \text{ for all } q \text{ such that } 1 \leq q \leq 2n+1. \\
\mathcal{T}(\mathsf{s}) \xrightarrow{i} \mathcal{T}(\mathsf{t}) \quad &\Longleftrightarrow \quad \text{for some } q \in \{1, 2, \ldots, k\} \text{ we have } \tau(\mathsf{s})_q + 1 = \tau(\mathsf{t})_q \text{ while} \\
&\qquad \tau(\mathsf{s})_p = \tau(\mathsf{t})_p \text{ for } p \neq q \text{ and} \\
&\qquad i = \begin{cases} q - \tau(\mathsf{t})_q + 2n + 1 - k & \text{if } q - \tau(\mathsf{t})_q \leq k - n - 1 \\ -(q - \tau(\mathsf{t})_q) + k & \text{if } q - \tau(\mathsf{t})_q > k - n - 1 \end{cases} \\
&\Longleftrightarrow \quad \text{for some } q \in \{1, 2, \ldots, k\} \text{ we have } T(\mathsf{s})_q - 1 = T(\mathsf{t})_q \text{ while} \\
&\qquad T(\mathsf{s})_p = T(\mathsf{t})_p \text{ for } p \neq q \text{ and} \\
&\qquad i = \begin{cases} q - \tau(\mathsf{t})_q + 2n + 1 - k & \text{if } q - \tau(\mathsf{t})_q \leq k - n - 1 \\ -(q - \tau(\mathsf{t})_q) + k & \text{if } q - \tau(\mathsf{t})_q > k - n - 1 \end{cases} \\
&\Longleftrightarrow \quad \mathsf{s}_{i+1} = \mathsf{t}_i = 1, \mathsf{s}_i = \mathsf{t}_{i+1} = 0, \text{ and } \mathsf{s}_p = \mathsf{t}_p \text{ for } p \notin \{i, i+1\}; \text{ or} \\
&\qquad \mathsf{s}_{2n+2-i} = \mathsf{t}_{2n+1-i} = 1, \mathsf{s}_{2n+1-i} = \mathsf{t}_{2n+2-i} = 0, \text{ and} \\
&\qquad \mathsf{s}_p = \mathsf{t}_p \text{ for } p \notin \{2n+1-i, 2n+2-i\}.
\end{aligned}$$

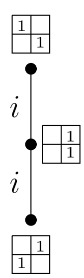
Proof. When $L = \widetilde{L}_{\mathsf{B}}(k, 2n+1-k)$, it is straightforward to check that the partial ordering of ideals is equivalent to the stated partial orderings of quasi-partitions, columnar tableaux, and tally diagrams. Edge colors in $\widetilde{L}_{\mathsf{B}}(k, 2n+1-k)$ are derived from the coloring of vertices in $\widetilde{P}_{\mathsf{B}}(k, 2n+1-k)$ (cf. Definition 3.1). Quasi-partitions inherit this edge-coloring in the obvious way (compare the rule for color i in the theorem statement with the definition of the vertex-coloring function in Definition 3.1). Proof details for the remaining claims about $\widetilde{L}_{\mathsf{B}}(k, 2n+1-k)$ in Proposition

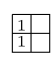
3.4 are routine. Since $\widetilde{P}_{\mathsf{B}}(k, 2n+1-k)$ is a weak subposet of $P^{KN}_{\mathsf{B}}(k, 2n+1-k)$ and $P^{DeC}_{\mathsf{B}}(k, 2n+1-k)$, it follows that $L^{KN}_{\mathsf{B}}(k, 2n+1-k)$ and $L^{DeC}_{\mathsf{B}}(k, 2n+1-k)$ are full-length sublattices of $\widetilde{L}_{\mathsf{B}}(k, 2n+1-k)$, so the claims of the proposition apply within these sublattices as well. $\square$

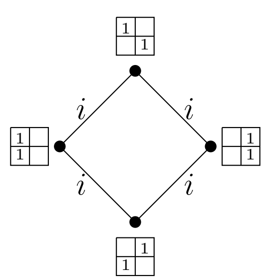
Theorem 3.5 *Let L be any one of $\widetilde{L}_{\mathsf{B}}(k, 2n+1-k)$, $L^{KN}_{\mathsf{B}}(k, 2n+1-k)$, or $L^{DeC}_{\mathsf{B}}(k, 2n+1-k)$. Then:*

(1) Let $i \in \{1, 2, \ldots, n-1\}$, thought of as an edge color in L. In the depictions of i-components immediately below, we only depict the i-slots of the tally diagram at any given vertex. Then any i-component of L is one of:

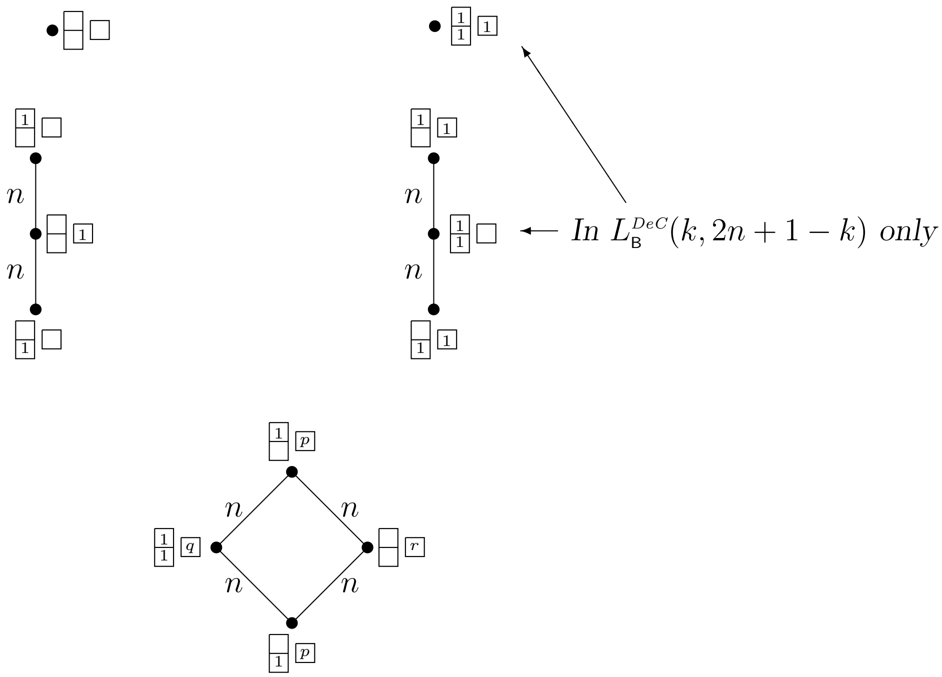




$\longleftarrow$ *In $L^{KN}_{\mathsf{B}}(k, 2n+1-k)$ only, when*  *is KN-inadmissible*



Now let $i = n$. In the depictions of n-components immediately below, we only depict the n-slots of the tally diagram at any given vertex. Then any n-component of L is one of:



The four-element n-component immediately above can only occur in

$\widetilde{L}_{\mathsf{B}}(k, 2n+1-k)$ or $L_{\mathsf{B}}^{KN}(k, 2n+1-k)$.

In this particular component, we have $q = p-1$ and $r = p+1$.

(2) Now let $\mathsf{t} = (\mathsf{t}_1, \ldots, \mathsf{t}_{2n+1})$ be an element of L. Then

$$wt(\mathsf{t}) = (\mathsf{t}_1 - \mathsf{t}_2 + \mathsf{t}_{2n} - \mathsf{t}_{2n+1}, \ldots, \mathsf{t}_i - \mathsf{t}_{i+1} + \mathsf{t}_{2n+1-i} - \mathsf{t}_{2n+2-i}, \ldots$$
$$\ldots, \mathsf{t}_{n-1} - \mathsf{t}_n + \mathsf{t}_{n+2} - \mathsf{t}_{n+3}, 2\mathsf{t}_n - 2\mathsf{t}_{n+2})$$

(3) L is B_n-structured.

(4) There exists a weight-preserving bijection $\phi : L_{\mathsf{B}}^{DeC}(k, 2n+1-k) \longrightarrow L_{\mathsf{B}}^{KN}(k, 2n+1-k)$, so that

$$\mathsf{WGF}(L_{\mathsf{B}}^{DeC}(k, 2n+1-k); z_1, z_2, \ldots, z_n) = \mathsf{WGF}(L_{\mathsf{B}}^{KN}(k, 2n+1-k); z_1, z_2, \ldots, z_n).$$

Moreover, $\mathsf{RGF}(L_{\mathsf{B}}^{DeC}(k, 2n+1-k); q) = \mathsf{RGF}(L_{\mathsf{B}}^{KN}(k, 2n+1-k); q) = \binom{2n+1}{k}_q$.

Proof. For part *(1)*, each of the sixteen possible i-slots can appear in one and only one of the i-components depicted above, when $1 \leq i < n$. Note that an i-component that is a length two chain can only occur when $L = L_{\mathsf{B}}^{KN}(k, 2n+1-k)$ and $\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 1 & 0 \\ \hline \end{array}$ is KN-inadmissible. Now consider $i = n$. There are exactly eight possibilities for the n-slots of any tally diagram in $L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$, and these are represented in the two singletons and the two three-element color n chains depicted above. The typical tally diagram from $\widetilde{L}_{\mathsf{B}}(k, 2n+1-k)$ or $L_{\mathsf{B}}^{KN}(k, 2n+1-k)$ has n-slots of the form $\begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array}\ \boxed{p}$, where p is a nonnegative integer and $a, b \in \{0, 1\}$. If $p \geq 2$ or if $a = 1 = b$ of if $p = 1$ with exactly one of a or b nonzero, then we must be in the four-element n-component depicted above. If $p = 1$ and $a = 0 = b$ or if $p = 0$ with exactly one of a or b nonzero, then we must be in the leftmost three-element color n chain depicted above. If $p = 0$ and $a = 0 = b$, then we must be in the leftmost of the two singletons depicted above. This exhausts all possibilities for a, b, and p, and thus completes our analysis of the $i = n$ case.

Now we prove part *(2)*. Since the i^{th} coordinate of $wt(\mathsf{t})$ is completely determined by the combinatorics of the i-component containing t, then it suffices to verify that $\rho_i(\mathsf{t}) - \delta_i(\mathsf{t})$ is the appropriate right-hand-side quantity from the identity claimed in *(2)*. This can be done easily by checking the cases depicted in part *(1)*.

To prove *(3)*, we note that it is enough to check that for any distinct edge colors i and j and any edge $\mathsf{s} \xrightarrow{i} \mathsf{t}$, we have

$$\rho_j(\mathsf{s}) - \delta_j(\mathsf{s}) + a_{ij} = \rho_j(\mathsf{t}) - \delta_j(\mathsf{t}), \tag{2}$$

where a_{ij} is the (i, j)-entry of the B_n Cartan matriix with rows/columns indexed in concert with the node labels of the (finitary) GCM graph B_n from Figure 2.1. Now, by part *(2)* of the theorem, for any tally diagram x we have $\rho_j(\mathsf{x}) - \delta_j(\mathsf{x}) = \mathsf{x}_j - \mathsf{x}_{j+1} + \mathsf{x}_{2n+1-j} - \mathsf{x}_{2n+2-j}$ for any $j \in \{1, 2, \ldots, n-1\}$ and $\rho_n(\mathsf{x}) - \delta_n(\mathsf{x}) = 2\mathsf{x}_n - 2\mathsf{x}_{n+2}$.

Begin with $1 \leq i < n$, and suppose i and j are adjacent nodes in the GCM graph B_n with $j = i+1$. Moreover, assume $j < n$ as well, so $a_{i,j} = -1 = a_{j,i}$. Then our edge $\mathsf{s} \xrightarrow{i} \mathsf{t}$ is one of the following, where we only depict the i-slots and the $(i+1)$-slots for each of s and t:

$$\mathsf{s} = \begin{array}{|c|c|c|}\hline \mathsf{s}_i & \mathsf{s}_{i+1} & \mathsf{s}_{i+2} \\ \hline \mathsf{s}_{2n+2-i} & \mathsf{s}_{2n+1-i} & \mathsf{s}_{2n-i} \\ \hline \end{array} \xrightarrow{i} \begin{array}{|c|c|c|}\hline \mathsf{s}_i + 1 & \mathsf{s}_{i+1} - 1 & \mathsf{s}_{i+2} \\ \hline \mathsf{s}_{2n+2-i} & \mathsf{s}_{2n+1-i} & \mathsf{s}_{2n-i} \\ \hline \end{array} = \mathsf{t}$$

or

$$\mathsf{s} = \begin{array}{|c|c|c|}\hline \mathsf{s}_i & \mathsf{s}_{i+1} & \mathsf{s}_{i+2} \\ \hline \mathsf{s}_{2n+2-i} & \mathsf{s}_{2n+1-i} & \mathsf{s}_{2n-i} \\ \hline \end{array} \xrightarrow{i} \begin{array}{|c|c|c|}\hline \mathsf{s}_i & \mathsf{s}_{i+1} & \mathsf{s}_{i+2} \\ \hline \mathsf{s}_{2n+2-i} - 1 & \mathsf{s}_{2n+1-i} + 1 & \mathsf{s}_{2n-i} \\ \hline \end{array} = \mathsf{t}.$$

In the former case, we have $\rho_{i+1}(\mathsf{s}) - \delta_{i+1}(\mathsf{s}) = \mathsf{s}_{i+1} - \mathsf{s}_{i+2} + \mathsf{s}_{2n-i} - \mathsf{s}_{2n+1-i}$ and $\rho_{i+1}(\mathsf{t}) - \delta_{i+1}(\mathsf{t}) = (\mathsf{s}_{i+1} - 1) - \mathsf{s}_{i+2} + \mathsf{s}_{2n-i} - \mathsf{s}_{2n+1-i}$. In the latter case, we have $\rho_{i+1}(\mathsf{s}) - \delta_{i+1}(\mathsf{s}) = \mathsf{s}_{i+1} - \mathsf{s}_{i+2} + \mathsf{s}_{2n-i} - \mathsf{s}_{2n+1-i}$ and $\rho_{i+1}(\mathsf{t}) - \delta_{i+1}(\mathsf{t}) = \mathsf{s}_{i+1} - \mathsf{s}_{i+2} + \mathsf{s}_{2n-i} - (\mathsf{s}_{2n+1-i} + 1)$. Either way, $\rho_{i+1}(\mathsf{s}) - \delta_{i+1}(\mathsf{s}) + a_{i,i+1} = \rho_{i+1}(\mathsf{t}) - \delta_{i+1}(\mathsf{t})$, thereby verifying identity (2) above when $j = i+1$. A similar argument establishes (2) under the supposition $j = i - 1$.

Suppose now that $i = n-1$ and $j = n$, so $a_{i,j} = a_{n-1,n} = -2$ and $a_{j,i} = a_{n,n-1} = -1$. Then our edge $\mathsf{s} \xrightarrow{i} \mathsf{t}$ is one of the following, where we only depict the $(n-1)$-slots and the n-slots for each of s and t:

$$\mathsf{s} = \begin{array}{|c|c|}\hline \mathsf{s}_{n-1} & \mathsf{s}_n \\ \hline \mathsf{s}_{n+3} & \mathsf{s}_{n+2} \\ \hline \end{array} \ \boxed{\mathsf{s}_{n+1}} \quad \xrightarrow{i} \quad \begin{array}{|c|c|}\hline \mathsf{s}_{n-1} + 1 & \mathsf{s}_n - 1 \\ \hline \mathsf{s}_{n+3} & \mathsf{s}_{n+2} \\ \hline \end{array} \ \boxed{\mathsf{s}_{n+1}} \quad = \mathsf{t}$$

or

$$\mathsf{s} = \begin{array}{|c|c|}\hline \mathsf{s}_{n-1} & \mathsf{s}_n \\ \hline \mathsf{s}_{n+3} & \mathsf{s}_{n+2} \\ \hline \end{array} \ \boxed{\mathsf{s}_{n+1}} \quad \xrightarrow{i} \quad \begin{array}{|c|c|}\hline \mathsf{s}_{n-1} & \mathsf{s}_n \\ \hline \mathsf{s}_{n+3} - 1 & \mathsf{s}_{n+2} + 1 \\ \hline \end{array} \ \boxed{\mathsf{s}_{n+1}} \quad = \mathsf{t}.$$

In the former case, we have $\rho_n(\mathsf{s}) - \delta_n(\mathsf{s}) = 2\mathsf{s}_n - 2\mathsf{s}_{n+2}$ and $\rho_n(\mathsf{t}) - \delta_n(\mathsf{t}) = 2(\mathsf{s}_n - 1) - 2\mathsf{s}_{n+2}$. In the latter case, we have $\rho_n(\mathsf{s}) - \delta_n(\mathsf{s}) = 2\mathsf{s}_n - 2\mathsf{s}_{n+2}$ and $\rho_n(\mathsf{t}) - \delta_n(\mathsf{t}) =$

$2\mathsf{s}_n - 2(\mathsf{s}_{n+2} + 1)$. Either way, $\rho_n(\mathsf{s}) - \delta_n(\mathsf{s}) + a_{n-1,n} = \rho_n(\mathsf{t}) - \delta_n(\mathsf{t})$. A similar argument establishes (2) under the supposition $j = n-1$ and $i = n$.

Finally, suppose i and j are distant nodes in the GCM graph for B_n, so $|j-i| > 1$. Then $a_{i,j} = 0 = a_{j,i}$ Now, t is formed from s by changing entries in the i-slots of s. Since the j-slots of s are distant from the i-slots, then t will have the exact same same j-slots as s. Then, $\rho_j(\mathsf{s}) - \delta_j(\mathsf{s}) + a_{ij} = \rho_j(\mathsf{s}) - \delta_j(\mathsf{s}) + 0 = \rho_j(\mathsf{s}) - \delta_j(\mathsf{s}) = \rho_j(\mathsf{t}) - \delta_j(\mathsf{t})$, completing the proof of *(3)*.

For *(4)*, it suffices to produce the claimed weight-preserving bijection, since the remaining claims of *(4)* will follow readily as consequences. To this latter point, note that equality of the weight generating functions for $L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$ and $L_{\mathsf{B}}^{KN}(k, 2n+1-k)$ will be an automatic consequence of the existence of the weight-preserving bijection. Since each lattice is B_n-structured by part *(3)* above, it follows from Lemma 3.4.2 of [Don7] that if two elements from one of these lattices have the same weight, then the elements have the same rank. Therefore,$L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$ and $L_{\mathsf{B}}^{KN}(k, 2n+1-k)$ will have the same rank generating function. Now, $L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$ is isomorphic to $L_{\mathsf{A}}(k, 2n+1-k)$, if edge colors are disregarded. Therefore $\mathsf{RGF}(L_{\mathsf{B}}^{DeC}(k, 2n+1-k); q) = \binom{2n+1}{k}_q$, so it will follow from the existence of a weight-preserving bijection that $\mathsf{RGF}(L_{\mathsf{B}}^{KN}(k, 2n+1-k); q) = \binom{2n+1}{k}_q$ also.

The weight-preserving bijection we construct next will match tally diagrams in corresponding "weight spaces" (to be defined shortly) of $L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$ and $L_{\mathsf{B}}^{KN}(k, 2n+1-k)$ respectively. But first an observation: For any given integer n-tuple μ, there exists a tally diagram $\mathsf{t} \in \widetilde{L}_{\mathsf{B}}(k, 2n+1-k)$ with $wt(\mathsf{t}) = \mu$ if and only if there exist tally diagrams $\mathsf{t}' \in L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$ and $\mathsf{t}'' \in L_{\mathsf{B}}^{KN}(k, 2n+1-k)$ such that $wt(\mathsf{t}') = wt(\mathsf{t}'') = \mu$. Fix such an integer n-tuple μ. The associated weight spaces of $L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$ and $L_{\mathsf{B}}^{KN}(k, 2n+1-k)$ are defined as follows:

$$\begin{aligned} \text{Weight}_{\mathsf{B}}^{DeC}(n,k;\mu) &:= \{\mathtt{x} \in L_{\mathsf{B}}^{DeC}(k, 2n+1-k) \,|\, wt(\mathtt{x}) = \mu\} \\ \text{Weight}_{\mathsf{B}}^{KN}(n,k;\mu) &:= \{\mathtt{x} \in L_{\mathsf{B}}^{KN}(k, 2n+1-k) \,|\, wt(\mathtt{x}) = \mu\} \end{aligned}$$

We will use a so-called reflection argument (often attributed to André cf. [Com]) to give a bijection from $\text{Weight}_{\mathsf{B}}^{DeC}(n,k;\mu)$ to $\text{Weight}_{\mathsf{B}}^{KN}(n,k;\mu)$. This requires that we identify weight space elements with paths in the coordinate plane. To this end, suppose μ has m zeros. Then tally diagrams in $\widetilde{L}_{\mathsf{B}}(k, 2n+1-k)$ with weight μ have exactly m full and empty slots, with at most $j := \lfloor \frac{k-n+m}{2} \rfloor \leq \lfloor \frac{m}{2} \rfloor$ full slots. Moreover, the positions of the full/empty slots of any two such tally diagrams $\mathtt{s}$ and $\mathtt{t}$ are the same (although the number of full slots in $\mathtt{s}$ need not be the same as the number of full slots in $\mathtt{t}$). We view a sequence of m full and empty slots as a path of unit steps in the coordinate plane as follows: Let an empty slot correspond to a unit step in the North direction and a full slot as a unit step East. Then the tally diagrams in $\text{Weight}_{\mathsf{B}}^{KN}(n,k;\mu)$ can be identified with the set of all paths that start at $(0,0)$, stay weakly above the line $y = x$ (i.e. such paths can touch but not cross $y = x$), and terminate at some point $(t, m-t)$, where $0 \leq t \leq j$; denote this latter set of paths by $\text{Path}_{\mathsf{B}}^{KN}(n,k;\mu)$. On the other hand, tally diagrams in $\text{Weight}_{\mathsf{B}}^{DeC}(n,k;\mu)$ have exactly j full slots, and so these correspond to the set of all paths from $(0,0)$ to $(j, m-j)$, a set we denote by $\text{Path}_{\mathsf{B}}^{DeC}(n,k;\mu)$. We will therefore match the paths in $\text{Path}_{\mathsf{B}}^{DeC}(n,k;\mu)$ to the paths in $\text{Path}_{\mathsf{B}}^{KN}(n,k;\mu)$. Now, those paths in $\text{Path}_{\mathsf{B}}^{DeC}(n,k;\mu)$ that stay weakly above $y = x$ correspond exactly to those paths in $\text{Path}_{\mathsf{B}}^{KN}(n,k;\mu)$ which have exactly j steps East. Now apply the André reflection principle to those paths in $\text{Path}_{\mathsf{B}}^{DeC}(n,k;\mu)$ that do *not* stay weakly above the line $y = x$: If $(p+1, p)$ is the first point below $y = x$ that such a path reaches, then swap all the preceding East and North moves to get a path from from $(1,-1)$ to $(j, m-j)$. This process yields all paths from $(1,-1)$ to $(j, m-j)$. Translate these north one unit and west one unit to get all paths from $(0,0)$ to $(j-1, m+1-j)$. Amongst these, the paths that stay weakly above $y=x$ correspond to the paths $\mathrm{Path}_{\mathsf{B}}^{KN}(n,k;\mu)$ with exactly $j-1$ steps East. Continue this procedure to obtain all paths in $\mathrm{Path}_{\mathsf{B}}^{KN}(n,k;\mu)$. This bijection of paths from $\mathrm{Path}_{\mathsf{B}}^{DeC}(n,k;\mu)$ to $\mathrm{Path}_{\mathsf{B}}^{DeC}(n,k;\mu)$ yields the desired bijection from $\mathrm{Weight}_{\mathsf{B}}^{DeC}(n,k;\mu)$ to $\mathrm{Weight}_{\mathsf{B}}^{KN}(n,k;\mu)$. $\square$

Chapter 4

A brief discourse on Weyl group symmetric functions

For a self-contained account of the foundations of Weyl symmetric function theory and Weyl bialternants, see [Don7], a tutorial that aims to synthesize and unify standard content from various classical sources (e.g. [FH], [Hum]). Here, we make note of some key notions. Our starting point is a finite rank n root system Φ residing in an n-dimensional Euclidean space $\mathfrak{E}$ with inner product $\langle\cdot,\cdot\rangle$. The related objects — coroots $\alpha^\vee := \frac{1}{\langle\alpha,\alpha\rangle}\alpha$; simple roots $\{\alpha_i\}$; Cartan matrix $A_\Phi := (a_{ij})_{i,j\in\{1,2,\ldots,n\}}$ with $a_{ij} := \langle\alpha_i,\alpha_j^\vee\rangle$; sets of positive and negative roots Φ^+ and Φ^- respectively; fundamental weights $\{\omega_i\}$ dual to the simple coroots $\{\alpha_j^\vee\}$ via the relations $\langle\omega_i,\alpha_j^\vee\rangle = \delta_{ij}$; the lattice of weights $\Lambda := \{\mu = \sum_{i=1}^n m_i\omega_i \,|\, m_i \in \mathbb{Z}\}$; dominant weights $\Lambda^+ = \{\mu = \sum_{i=1}^n m_i\omega_i \,|\, m_i \in \mathbb{Z} \text{ with } m_i \geq 0\}$; finite Weyl group W with generators $\{s_i\}_{i\in\{1,2,\ldots,n\}}$ and relations $(s_is_j)^{m_{ij}} = \varepsilon$ if $m_{ij} = k \in \{1,2,3,4,6\}$ with k as the smallest positive integer such that $a_{ij}a_{ji} = 4\cos^2(\pi/k)$; action of W on Λ given by $s_i.\mu = \mu - \langle\mu,\alpha_i^\vee\rangle\alpha_i$ for each $i \in \{1,\ldots,n\}$ and each $\mu \in \Lambda$; special elements $\varrho := \sum\omega_i = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ and $\varrho^\vee = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha^\vee$; etc — are obtained as usual. The (finite) irreducible root systems are classified by the finitary GCM graphs of Figure 2.1.

For an extended example, including explicit computations and illustrations of most all of the notions of the preceding paragraph, see Chapter 2 subsection 2.18 (p.48) of [Gil]. However, a reader can safely ignore the Weyl group / Lie representation theoretic gadgetry here if their interests are mainly combinatorial. In this case, such a reader can think of the main combinatorial goal as that of realizing the conclusions of Proposition 4.4 for our type B elementary lattices by establishing the combinatorial criteria of Theorem 4.1.

The group ring $\mathbb{Z}[\Lambda]$ has as a $\mathbb{Z}$-basis the formal exponentials $\{e^{\mu}\}_{\mu\in\Lambda}$. The Weyl group W acts on $\mathbb{Z}[\Lambda]$ via $s_i.e^{\mu} := e^{s_i\cdot\mu}$. From here on, we identify each e^{ω_i} as the indeterminate z_i. Then for $\mu = \sum m_i\omega_i \in \Lambda$, the quantity $e^{\mu} = e^{m_1\omega_1+m_2\omega_2+\cdots+m_n\omega_n}$ is the monomial $z_1^{m_1}z_2^{m_2}\cdots z_n^{m_n}$. That is, each $\chi \in \mathbb{Z}[\Lambda]$ is a Laurent polynomial in the variables $z_1, z_2, \ldots, z_n$. The *ring of Weyl symmetric functions* $\mathbb{Z}[\Lambda]^W$ is the subring of W-invariants in $\mathbb{Z}[\Lambda]$. Each $\chi \in \mathbb{Z}[\Lambda]^W$ is a *Weyl symmetric function*; at times for clarity we use "Φ" as a modifier, as in "type Φ Weyl symmetric function" or "Φ-symmetric function." The *subgroup of W-alternants* $\mathbb{Z}[\Lambda]^{\text{alt}}$ consists of those group ring elements φ for which $\sigma.\varphi = \det(\sigma)\varphi$ for all Weyl group elements σ. Define a mapping $\mathcal{A} : \mathbb{Z}[\Lambda] \longrightarrow \mathbb{Z}[\Lambda]^{\text{alt}}$ by the rule $\mathcal{A}(\varphi) := \sum_{\sigma\in W}\det(\sigma)\sigma.\varphi$. The Weyl denominator is the alternant $\mathcal{A}(e^{\varrho})$, which factors as

$$\mathcal{A}(e^{\varrho}) = e^{\varrho}\Big(\prod_{\alpha\in\Phi^+}(1-e^{-\alpha})\Big) = \prod_{\alpha\in\Phi^+}(e^{\alpha/2}-e^{-\alpha/2}) = e^{-\varrho}\Big(\prod_{\alpha\in\Phi^+}(e^{\alpha}-1)\Big).$$

The following is a sort of fundamental theorem for Weyl symmetric functions: For any dominant weight λ, there exists a unique $\chi_{\lambda}^{\Phi} \in \mathbb{Z}[\Lambda]$ for which $\mathcal{A}(e^{\varrho})\,\chi_{\lambda}^{\Phi} = \mathcal{A}(e^{\lambda+\varrho})$, and in fact the χ_{λ}^{Φ}'s comprise a $\mathbb{Z}$-basis for $\mathbb{Z}[\Lambda]^W$. Each χ_{λ}^{Φ} is a *Weyl bialternant**.

*Let $\mathfrak{g}(\Phi)$ be the finite-dimensional semisimple complex Lie algebra affiliated with Φ by, say, a construction using generators and relations. Each finite-dimensional irreducible representation V of $\mathfrak{g}(\Phi)$ is uniquely identified by the dominant weight λ of a highest weight basis vector. The Weyl character formula connects V with χ_{λ}^{Φ} via the formula

$$\sum_{\mu\in\Lambda} V_{\mu} = \chi_{\lambda}^{\Phi},$$

The symmetric functions $\{\chi^{\mathsf{A}_{N-1}}_{\omega_k}\}_{k\in\{1,\ldots,N-1\}}$ are (after a change of variables) the classical elementary symmetric functions. By analogy, for our generic (finite) root system Φ, we say the *elementary Φ-symmetric functions* are $\{\chi^{\Phi}_{\omega_k}\}_{k\in\{1,\ldots,n\}}$ In the above language, our goal is to find poset models for the elementary B_n-symmetric functions $\{\chi^{\mathsf{B}_n}_{\omega_k}\}_{k\in\{1,\ldots,n-1\}}$ together with what we call the *almost-elementary* B_n-symmetric function $\chi^{\mathsf{B}_n}_{2\omega_n}$. From here on, let $\mathbf{c} : \{1, 2, \ldots, n\} \longrightarrow \{1, 2\}$ be the function for which

$$\mathbf{c}(k) = 1 + \lfloor \frac{k}{n} \rfloor.$$

Next we define precisely what we mean by a "poset model" for a Weyl bialternant $\chi^{\mathsf{X}_n}_{\lambda}$, where $\mathsf{X} \in \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \mathsf{F}, \mathsf{G}\}$. If R is X_n-structured and if it weight generating function $\mathsf{WGF}(R; z_1, z_2, \ldots, z_n) = \chi^{\mathsf{X}_n}_{\lambda}$, we say R is a *splitting poset for* $\chi^{\mathsf{X}_n}_{\lambda}$. If R is a modular (respectively distributive) lattice, then call R a *splitting modular* (resp. *distributive*) *lattice*. Expressed in this terminology, the aim of this paper is to show that for integers k and n with $1 \le k \le n$, we have

$$\mathsf{WGF}\Big(L^{DeC}_{\mathsf{B}}(k, 2n+1-k); z_1, z_2, \ldots, z_n\Big) = \mathsf{WGF}\Big(L^{KN}_{\mathsf{B}}(k, 2n+1-k); z_1, z_2, \ldots, z_n\Big) = \chi^{\mathsf{B}_n}_{\mathbf{c}(k)\omega_k},$$

so that each of the B_n-structured lattices $L^{DeC}_{\mathsf{B}}(k, 2n+1-k)$ and $L^{KN}_{\mathsf{B}}(k, 2n+1-k)$ is a splitting distributive lattice for the k^{th} elementary B_n-symmetric function (when $k < n$) and the almost-elementary B_n-symmetric function (when $k = n$).

With the remainder of this chapter, we record criteria developed by Donnelly in [Don7] that will be applied in §5 to establish the preceding splitting distributive lattice claims, and we explore some consequences. Towards the end of this chapter, we demonstrate these ideas for the elementary A_{N-1}-symmetric functions and the elementary B_n-symmetric function $\chi^{\mathsf{B}_n}_{\omega_n}$.

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where V_μ is the subspace of V consisting of all weight vectors with weight μ. When $\lambda = \omega_k$, the irreducible representation is the k^{th} fundamental representation, and the weight ω_k is a fundamental weight.

Before we state our next theorem, we need some definitions/notation. In a product $\mathcal{C} := \mathcal{C}_1 \times \cdots \times \mathcal{C}_p$ of chains $\mathcal{C}_1, \ldots, \mathcal{C}_p$, a *subface* is a set $\mathcal{S}_q = \{(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p) \in \mathcal{C} \,|\, \mathbf{x}_q \text{ is not maximal in } \mathcal{C}_q\}$, for some fixed $q \in \{1, \ldots, p\}$. If a poset P is isomorphic to $\mathcal{C}$ via some isomorphism $\varphi : P \longrightarrow \mathcal{C}$, then a *$\varphi$-subface* of P is a set $\mathcal{S} = \varphi^{-1}(\mathcal{S}_q)$ for some $q \in \{1, \ldots, p\}$; if the isomorphism φ is understood, we simply call $\mathcal{S}$ a subface of P. Now suppose R is a ranked poset whose edges are colored by the set $\{1, 2, \ldots, n\}$, and say $\mathcal{M}$ is a vertex subset of R. A vertex-coloring function $\kappa : R \setminus \mathcal{M} \longrightarrow \{1, 2, \ldots, n\}$ is *subface-friendly* if, for all $\mathbf{x} \in R \setminus \mathcal{M}$, the set

$$\Big\{\mathbf{y} \in \mathbf{comp}_{\kappa(\mathbf{x})}(\mathbf{x}) \Big|\, \mathbf{y} \notin \mathcal{M} \text{ and } \kappa(\mathbf{y}) = \kappa(\mathbf{x})\Big\}$$

is a subface of $\mathbf{comp}_{\kappa(\mathbf{x})}(\mathbf{x})$.

Theorem 4.1 *Let X_n be an irreducible rank n root system, where X is one of $\{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \mathsf{F}, \mathsf{G}\}$. Suppose R is an X_n-structured poset with a unique maximal element $\mathbf{m}$, so $\lambda := wt(\mathbf{m})$ is dominant in Λ. Further suppose that for each $i \in \{1, 2, \ldots, n\}$, each i-component of R is isomorphic to a product of chains. Finally, suppose there exists a subface-friendly vertex-coloring function $\kappa : R \setminus \{\mathbf{m}\} \longrightarrow \{1, 2, \ldots, n\}$. Then R is a splitting poset for $\chi_\lambda^{\mathsf{X}_n}$, i.e.*

$$\mathsf{WGF}\Big(R; z_1, z_2, \ldots, z_n\Big) = \chi_\lambda^{\mathsf{X}_n}.$$

The following simple corollary provides criteria that simplify the application of the preceding theorem in some circumstances.

Corollary 4.2 *With X_n as in Theorem 4.1, suppose R is an X_n-structured poset with a unique maximal element $\mathbf{m}$, so $\lambda := wt(\mathbf{m})$ is dominant in Λ. Suppose that for each $i \in \{1, 2, \ldots, n\}$, each i-component of R is a chain with either one or two elements. Then R is a splitting poset for $\chi_\lambda^{\mathsf{X}_n}$.*

Proof. Let $\mathbf{x} \in R \setminus \{\mathbf{m}\}$. Then $\{\mathbf{y} \in R \,|\, \mathbf{x} \rightarrow \mathbf{y}\}$ is nonempty. Choose some $\mathbf{y}(\mathbf{x})$ with $\mathbf{x} \rightarrow \mathbf{y}(\mathbf{x})$ and let i be the color of this edge. Set $\kappa(\mathbf{x}) := i$. Clearly the set

$$\left\{\mathbf{y} \in \mathbf{comp}_i(\mathbf{x}) \middle| \, \mathbf{y} \neq \mathbf{m} \text{ and } \kappa(\mathbf{y}) = i\right\} = \{\mathbf{x}\}$$

is a subface of $\mathbf{comp}_i(\mathbf{x}) = \{\mathbf{x} \stackrel{i}{\rightarrow} \mathbf{y}(\mathbf{x})\}$. Then the vertex-coloring function $\kappa : R \setminus \{\mathbf{m}\} \longrightarrow \{1, 2, \ldots, n\}$ is subface-friendly. Now apply Theorem 4.1. □

The next result showcases some pleasant enumerative and combinatorial-structural aspects of any connected splitting poset for a given Weyl bialternant. It is a simple poset-theoretic interpretation of some well-known Lie representation / Weyl symmetric function theoretic quantities; see Proposition 4.7 of [Don7] for a proof that utilizes the language and notation of this paper. Unimodality of the rank generating function is the only aspect of this result that depends upon the representation theory of semisimple Lie algebras, see for example Corollary 2.22 of [Don7].

Proposition 4.3 *Let* X_n *be an irreducible rank* n *root system, where* X is one of $\{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \mathsf{F}, \mathsf{G}\}$. *Let* λ *be dominant in* Λ*, and suppose* R *is a connected splitting poset for* $\chi_\lambda^{\mathsf{X}_n}$. *Then* R *has a unique rank function. Moreover,* R *is rank symmetric and rank unimodal, and its rank generating function is a polynomial that can be written as a quotient of products as follows, where the quantities* $\langle \lambda + \varrho, \alpha^\vee \rangle$ *and* $\langle \varrho, \alpha^\vee \rangle$ *are positive integers for each* $\alpha \in \Phi^+$*:*

$$\mathsf{RGF}(R; q) = \frac{\displaystyle\prod_{\alpha \in \Phi^+} \left(1 - q^{\langle \lambda + \varrho, \alpha^\vee \rangle}\right)}{\displaystyle\prod_{\alpha \in \Phi^+} \left(1 - q^{\langle \varrho, \alpha^\vee \rangle}\right)}.$$

We therefore obtain the following formulas for $\mathsf{CARD}(R)$ *and* $\mathsf{LENGTH}(R)$*:*

$$\mathsf{CARD}(R) = \frac{\displaystyle\prod_{\alpha \in \Phi^+} \langle \lambda + \varrho, \alpha^\vee \rangle}{\displaystyle\prod_{\alpha \in \Phi^+} \langle \varrho, \alpha^\vee \rangle}$$

$$\textit{and } \mathsf{LENGTH}(R) = \sum_{\alpha\in\Phi^+} \langle \lambda + \varrho, \alpha^\vee \rangle = 2\langle \lambda, \varrho^\vee \rangle. \qquad \square$$

For the the elementary A_{N-1}-symmetric functions and the elementary/almost-elementary B_n-symmetric functions, the data of Proposition 4.3 specializes as follows. This well-known data was obtained by employing methods appearing in Chapters 2 and 4 of [Gil]; see also Proposition D.23 of [Don7].

Proposition 4.4

$\boxed{\text{Type } \mathsf{A}_{N-1}}$ *For integers k and N with $1 \leq k \leq N-1$, let R be a connected splitting poset for $\chi^{\mathsf{A}_{N-1}}_{\omega_k}$. Then R is rank symmetric and rank unimodal. Moreover*

$$\mathsf{RGF}(R;q) = \binom{N}{k}_q, \quad \mathsf{CARD}(R) = \binom{N}{k}, \textit{ and } \mathsf{LENGTH}(R) = k(N-k).$$

$\boxed{\text{Type } \mathsf{B}_n}$ *For integers k and n with $n \geq 2$ and $1 \leq k \leq n$, let R be any connected splitting poset for $\chi^{\mathsf{B}_n}_{\mathbf{c}(k)\omega_k}$. Then R is rank symmetric and rank unimodal. Moreover*

$$\mathsf{RGF}(R;q) = \binom{2n+1}{k}_q, \quad \mathsf{CARD}(R) = \binom{2n+1}{k}, \textit{ and } \mathsf{LENGTH}(R) = k(2n+1-k).$$

Now let R be a connected splitting poset for $\chi^{\mathsf{B}_n}_{\omega_n}$. Then R is rank symmetric and rank unimodal. Moreover

$$\mathsf{RGF}(R;q) = \prod_{i=1}^{n}(1+q^i), \quad \mathsf{CARD}(R) = 2^n, \textit{ and } \mathsf{LENGTH}(R) = n(n+1)/2. \qquad \square$$

The next result is known, but now follows easily from Corollary 4.2 and Proposition 4.4 together with Proposition 2.2 and Proposition 2.4.

Theorem 4.5 *Let k, n, and N be integers, with $n \geq 2$, $N \geq 2$, and $1 \leq k \leq N-1$. Then the diamond-colored lattice $L_{\mathsf{A}}(k, N-k)$ is a splitting distributive lattice for $\chi^{\mathsf{A}_{N-1}}_{\omega_k}$. So, $L_{\mathsf{A}}(k, N-k)$ is rank symmetric and rank unimodal, and $\mathsf{RGF}(L_{\mathsf{A}}(k, N-k); q) = \binom{N}{k}_q$. Also, the diamond-colored lattice $L^{\mathrm{spin}}_{\mathsf{B}}(n)$ is a splitting distributive lattice for $\chi^{\mathsf{B}_n}_{\omega_n}$. Thus, $L^{\mathrm{spin}}_{\mathsf{B}}(n)$ is rank symmetric and rank unimodal, and $\mathsf{RGF}(L^{\mathrm{spin}}_{\mathsf{B}}(n); q) = \prod_{i=1}^{n}(1+q^i)$.*

Proof. By Proposition 2.2, $L_{\mathsf{A}}(k, N-k)$ is A_{N-1}-structured, and any i-component $(1 \leq i \leq N-1)$ is a one- or two-element chain. Observe that the unique maximal element has weight ω_k. So, by Corollary 4.2, $L_{\mathsf{A}}(k, N-k)$ is a splitting distributive lattice for $\chi_{\omega_k}^{\mathsf{A}_{N-1}}$. The other claims about $L_{\mathsf{A}}(k, N-k)$ in the theorem statement follow from Proposition 4.4. In an entirely similar way, the claims made about $L_{\mathsf{B}}^{\text{spin}}(n)$ follow from Proposition 2.4, Corollary 4.2, and Proposition 4.4. $\square$

In the next chapter, we use Theorem 4.1 together with Theorem 3.5 to establish that the type B elementary lattices $L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$ and $L_{\mathsf{B}}^{KN}(k, 2n+1-k)$ are splitting distributive lattices for the Weyl bialternant $\chi_{\mathbf{c}(k)\omega_k}^{\mathsf{B}_n}$.

Chapter 5

Type B elementary lattices as splitting posets

Our main goal in this chapter is to use the vertex-coloring method prescribed by Theorem 4.1 to prove the following theorem:

Theorem 5.1 *Fix integers n and k with $n \geq 2$ and $1 \leq k \leq n$. Let L be one of $L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$ or $L_{\mathsf{B}}^{KN}(k, 2n+1-k)$. Then L is a splitting distributive lattice for the the k^{th} type B_n elementary/almost elementary symmetric function. In particular,*

$$\mathsf{WGF}(L; z_1, ..., z_n) = \chi^{\mathsf{B}_n}_{\mathbf{c}(k)\omega_k}.$$

Before presenting the proof to the above theorem, we record the following corollary. Now, $L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$ is the same distributive lattice as $L_{\mathsf{A}}(k, 2n+1-k)$ when we ignore edge colors. Thus, the claims made about $L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$ in the next corollary can be proven directly (see [O], [Proc1], [Zeil]) and without reference to Weyl symmetric function theory; the same claims about $L_{\mathsf{B}}^{KN}(k, 2n+1-k)$ follow from Theorem 3.5.4. However, in view of Theorem 5.1 above, the corollary follows immediately from Proposition 4.4.

Corollary 5.2 *Keep the hypotheses of Theorem 5.1. Then L is rank symmetric and rank unimodal, and* $\mathsf{RGF}(L_{\mathsf{B}}^{DeC}(k, 2n+1-k); q) = \mathsf{RGF}(L_{\mathsf{B}}^{KN}(k, 2n+1-k); q) = \binom{2n+1}{k}_q$.

□

Proof of Theorem 5.1. Our proof only addresses $L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$. That $L_{\mathsf{B}}^{KN}(k, 2n+1-k)$ is a splitting distributive lattice for $\chi_{\mathbf{c}(k)\omega_k}^{\mathsf{B}_n}$ then follows from Theorem 3.5 parts *(3)* and *(4)*. In order to apply Theorem 4.1 and conclude that $L := L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$ is a splitting distributive lattice for $\chi_{\mathbf{c}(k)\omega_k}^{\mathsf{B}_n}$, we must verify that L possesses three combinatorial properties stated in that theorem as sufficient conditions. First, we must verify that the i-components of L are isomorphic to products of chains. This can be done by inspecting the classification of i-components in Theorem 3.5 part *(1)*. Second, we must verify that L is B_n-structured. But this was done in Theorem 3.5 part *(3)*. Third, we must verify that L has a subface-friendly vertex-coloring function

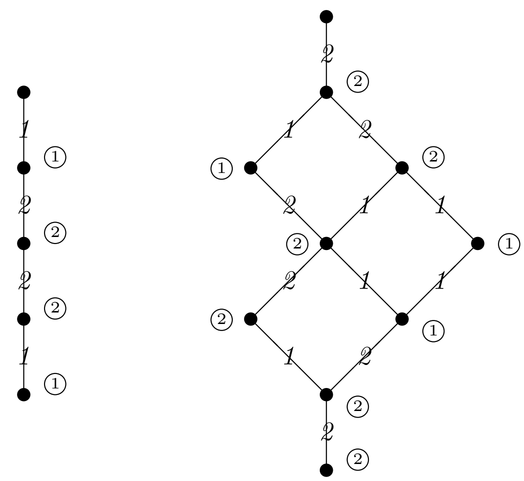
$$\kappa : L \setminus \{\mathbf{max}_{n,k}\} \to \{1, 2, ..., n\},$$

with $\mathbf{max}_{n,k}$ as the unique maximal element of L and where we abuse notation by identifying L with its vertex set $\mathcal{V}(L)$. We will prove the existence of such a function by induction on n. We note that the recursive process used to produce this function is constructive.

To set up the argument, we need some definitions and notation. For integers a and b, let $[a, b]$ denote the set of integers $\{a, a+1, \ldots, b-1, b\}$ when $a \leq b$; otherwise $[a, b] := \emptyset$. A B_{n-1}*-component* of L is just a $[2, n]$-component of L, and a B_{n-1}*-maximal element* is the unique maximal element of a given B_{n-1}-component. We can study B_{n-1}-components and their associated B_{n-1}-maximal elements using tally diagrams and by focussing in particular on the movement of tallies between slots 1 and 2. The next observation is crucial for our induction argument. Observe that, for $n \geq 3$, the B_{n-1}-components of $L_{\mathsf{B}}^{DeC}(k, 2n+1-k)$ are type B_{n-1} De Concini elementary lattices, in the following sense: If such a component $\mathcal{X}$ with maximal element $\mathfrak{x}$ is not a singleton, then $\mathcal{X}$ is naturally isomorphic to some $L_{\mathsf{B}}^{DeC}(j, 2n-1-j)$ (with

$1 \le j \le n-1$) where the colors $[2, n]$ of $\mathcal{X}$ are matched to the colors $[1, n-1]$ of $L_{\mathsf{B}}^{DeC}(j, 2n-1-j)$ via the set mapping $\sigma_n : [2, n] \longrightarrow [1, n-1]$ wherein $\sigma_n(x) = x-1$. We denote the preceding isomorphism by $\phi_{\mathsf{X}} : \mathcal{X} \longrightarrow L_{\mathsf{B}}^{DeC}(j, 2n-1-j)$.

For the basis step of our induction argument, we take n to be 2 and produce subface-friendly vertex-coloring functions for the following diamond-colored distributive lattices:



In the above pictures, the circled number near any given vertex represents the color of that vertex. By inspection we see that these vertex-coloring functions are subface-friendly.

For the induction step, we take an integer m with $2 \le m < n$ and assume that for any given integer j such that $1 \le j \le m$, there is a subface-friendly vertex-coloring function $\kappa_{m,j} : L_{\mathsf{B}}^{DeC}(j, 2m+1-j) \setminus \{\mathbf{max}_{m,j}\} \longrightarrow [1, m]$. We will produce a subface-friendly vertex-coloring function $\kappa_{m+1,j} : L_{\mathsf{B}}^{DeC}(j, 2m+3-j) \setminus \{\mathbf{max}_{m+1,j}\} \longrightarrow \{1, m+1\}$ whenever j is an integer with $1 \le j \le m+1$. To do so, we consider cases: $j = 1$, $j = 2$, $3 \le j \le m$, and $j = m+1$.

When $j = 1$, there are exactly three B_m-maximal elements in $L_{\mathsf{B}}^{DeC}(1, 2m+2)$:

$$\mathfrak{a} = \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & \cdots & 0 & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 \\ \hline \end{array}\ \boxed{0}\ , \ \mathfrak{b} = \begin{array}{|c|c|c|c|c|}\hline 0 & 0 & \cdots & 0 & 0 \\ \hline 1 & 0 & \cdots & 0 & 0 \\ \hline \end{array}\ \boxed{0}\ , \text{ and } \mathfrak{c} = \begin{array}{|c|c|c|c|c|}\hline 1 & 0 & \cdots & 0 & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 \\ \hline \end{array}\ \boxed{0}\ .$$

Tally diagram $\mathfrak{c}$ is the unique maximal element $\mathbf{max}_{m+1,1}$ of $L_{\mathsf{B}}^{DeC}(1, 2m+2)$, and its B_m-component is a singleton vertex; $\mathfrak{b}$ is the unique minimal element, and its B_m-component is also a singleton. Tally diagram $\mathfrak{a}$ is the maximal element in a B_m-component isomorphic to $L_{\mathsf{B}}^{DeC}(1, 2m)$ via the isomorphism $\phi_{\mathfrak{a}}$ in conjunction with the edge-color-matching function $\sigma_{m+1} : [2, m+1] \longrightarrow [1, m]$. We define a vertex-coloring function $\kappa_{m+1,1} : L_{\mathsf{B}}^{DeC}(1, 2m) \setminus \{\mathbf{max}_{m+1,1}\} \longrightarrow [1, m+1]$ as follows:

$$\kappa_{m+1,1}(\mathfrak{x}) = \begin{cases} 1 & \text{if } \mathfrak{x} = \mathfrak{a} \text{ or } \mathfrak{x} = \mathfrak{b} \\ \sigma_{m+1}^{-1}\Big((\kappa_{m,1} \circ \phi_{\mathfrak{a}})(\mathfrak{x})\Big) & \text{if } \mathfrak{x} \in \mathbf{comp}_{[2,m+1]}(\mathfrak{a}) \text{ and } \mathfrak{x} \neq \mathfrak{a} \end{cases}$$

Now, by our induction hypothesis, $\kappa_{m,1}$ is subface-friendly. So to prove that $\kappa_{m+1,1}$ is subface-friendly, it suffices to check that $\{\mathfrak{y} \in \mathbf{comp}_1(\mathfrak{x}) \,|\, \kappa_{m+1,1}(\mathfrak{y}) = 1\}$ is a subface of $\mathbf{comp}_1(\mathfrak{x})$ for $\mathfrak{x} \in \{\mathfrak{a}, \mathfrak{b}\}$. But this latter condition is trivial since, for each $\mathfrak{x} \in \{\mathfrak{a}, \mathfrak{b}\}$, we see that $\mathbf{comp}_1(\mathfrak{x})$ is a two-element chain, $\mathfrak{x}$ is its minimal element, and $\kappa_{m+1,1}(\mathfrak{y}) > 1$ if $\mathfrak{y} \in \mathbf{comp}_1(\mathfrak{x}) \setminus \{\mathfrak{x}\}$ and $\mathfrak{y} \neq \mathbf{max}_{m+1,1}$.

Next assume $j = 2$. In this case, there are exactly four B_m-maximal elements in $L_{\mathsf{B}}^{DeC}(2, 2m+1)$:

$$\mathfrak{a} = \begin{array}{|c|c|c|c|c|c|}\hline 0 & 1 & 1 & \cdots & 0 & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 \\ \hline \end{array}\ \boxed{0}\ , \ \mathfrak{b} = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 0 & \cdots & 0 & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 \\ \hline \end{array}\ \boxed{0}\ ,$$

$$\mathfrak{c} = \begin{array}{|c|c|c|c|c|c|}\hline 0 & 1 & 0 & \cdots & 0 & 0 \\ \hline 1 & 0 & 0 & \cdots & 0 & 0 \\ \hline \end{array}\ \boxed{0}\ , \text{ and } \mathfrak{d} = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 0 & 0 & \cdots & 0 & 0 \\ \hline 1 & 0 & 0 & \cdots & 0 & 0 \\ \hline \end{array}\ \boxed{0}\ .$$

The B_m-component of tally diagram $\mathfrak{a}$ is isomorphic to $L_{\mathsf{B}}^{DeC}(2, 2m-1)$. The B_m-components of tally diagrams $\mathfrak{b}$ and $\mathfrak{c}$ are isomorphic to $L_{\mathsf{B}}^{DeC}(1, 2m)$, and $\mathfrak{b}$ is the unique maximal element $\mathbf{max}_{m+1,2}$ of $L_{\mathsf{B}}^{DeC}(2, 2m+1)$. The B_m-component of tally diagram $\mathtt{d}$ is a singleton. Now define a vertex-coloring function $\kappa_{m+1,2} : L^{DeC}_{\mathsf{B}}(2, 2m+1) \setminus \{\mathbf{max}_{m+1,2}\} \longrightarrow [1, m+1]$ as follows:

$$\kappa_{m+1,2}(\mathtt{x}) = \begin{cases} 1 & \text{if } \mathtt{x} \in \{\mathtt{a}, \mathtt{c}, \mathtt{d}\} \\ \sigma^{-1}_{m+1}\Big((\kappa_{m,2} \circ \phi_{\mathtt{a}})(\mathtt{x})\Big) & \text{if } \mathtt{x} \in \mathbf{comp}_{[2,m+1]}(\mathtt{a}) \text{ and } \mathtt{x} \neq \mathtt{a} \\ \sigma^{-1}_{m+1}\Big((\kappa_{m,1} \circ \phi_{\mathtt{b}})(\mathtt{x})\Big) & \text{if } \mathtt{x} \in \mathbf{comp}_{[2,m+1]}(\mathtt{b}) \text{ and } \mathtt{x} \neq \mathtt{b} \\ \sigma^{-1}_{m+1}\Big((\kappa_{m,1} \circ \phi_{\mathtt{c}})(\mathtt{x})\Big) & \text{if } \mathtt{x} \in \mathbf{comp}_{[2,m+1]}(\mathtt{c}) \text{ and } \mathtt{x} \neq \mathtt{c} \end{cases}$$

As in the $j = 1$ case, we know by our induction hypothesis that $\kappa_{m,1}$ and $\kappa_{m,2}$ are subface-friendly. So to prove that $\kappa_{m+1,2}$ is subface-friendly, it suffices to check that $\{\mathtt{y} \in \mathbf{comp}_1(\mathtt{x}) \,|\, \kappa_{m+1,2}(\mathtt{y}) = 1\}$ is a subface of $\mathbf{comp}_1(\mathtt{x})$ for $\mathtt{x} \in \{\mathtt{a}, \mathtt{c}, \mathtt{d}\}$. For $\mathtt{x} = \mathtt{a}$, this latter condition is trivial since $\mathbf{comp}_1(\mathtt{a})$ is a two-element chain, $\mathtt{a}$ is its minimal element, and $\kappa_{m+1,2}(\mathtt{y}) > 1$ if $\mathtt{y} \in \mathbf{comp}_1(\mathtt{a}) \setminus \{\mathtt{a}\}$. Now note that $\mathbf{comp}_1(\mathtt{c}) = \mathbf{comp}_1(\mathtt{d})$ is a four-element diamond for which $\kappa_{m+1,2}(\mathtt{y}) > 1$ if $\mathtt{y} \in \mathbf{comp}_1(\mathtt{c}) \setminus \{\mathtt{c}, \mathtt{d}\}$. So, $\{\mathtt{y} \in \mathbf{comp}_1(\mathtt{x}) \,|\, \kappa_{m+1,2}(\mathtt{y}) = 1\} = \{\mathtt{c}, \mathtt{d}\}$ is a subface of $\mathbf{comp}_1(\mathtt{x})$ whenever $\mathtt{x} \in \{\mathtt{c}, \mathtt{d}\}$.

Now assume $3 \leq j \leq m$. As in the preceding case, there are exactly four B_m-maximal elements in $L^{DeC}_{\mathsf{B}}(j, 2m+3-j)$:

$$\mathtt{a} = \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & \cdots & 0 & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 \\ \hline \end{array}\ \boxed{0}\ , \quad \mathtt{b} = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & \cdots & 0 & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 \\ \hline \end{array}\ \boxed{0}\ ,$$

$$\mathtt{c} = \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & \cdots & 0 & 0 \\ \hline 1 & 0 & \cdots & 0 & 0 \\ \hline \end{array}\ \boxed{0}\ , \text{ and } \mathtt{d} = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & \cdots & 0 & 0 \\ \hline 1 & 0 & \cdots & 0 & 0 \\ \hline \end{array}\ \boxed{0}\ .$$

The ellipses on the top row of each of the above tally diagrams indicate a sequence of consecutive 1's beginning in position 3; the bottom row ellipses indicate empty boxes. The B_m-component of tally diagram $\mathtt{a}$ is isomorphic to $L^{DeC}_{\mathsf{B}}(j, 2m+1-j)$. The B_m-components of tally diagrams $\mathtt{b}$ and $\mathtt{c}$ are isomorphic to $L^{DeC}_{\mathsf{B}}(j-1, 2m+2-j)$, and $\mathtt{b} = \mathbf{max}_{m+1,j}$ is the unique maximal element of $L^{DeC}_{\mathsf{B}}(j, 2m+3-j)$. The B_m-component of tally diagram $\mathtt{d}$ is isomorphic to $L^{DeC}_{\mathsf{B}}(j-2, 2m+3-j)$. A subtle difference between the preceding case and this one is that $\mathbf{comp}_1(\mathtt{c})$ is a diamond consisting of the following four elements, none of which is $\mathtt{d}$:

$$\mathtt{c} = \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & \cdots & 0 & 0 \\ \hline 1 & 0 & \cdots & 0 & 0 \\ \hline \end{array} \ \boxed{0}\ \textbf{,}\ \mathtt{c}' = \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & \cdots & 0 & 0 \\ \hline 0 & 1 & \cdots & 0 & 0 \\ \hline \end{array} \ \boxed{0}\ \textbf{,}$$

$$\mathtt{c}'' = \begin{array}{|c|c|c|c|c|}\hline 1 & 0 & \cdots & 0 & 0 \\ \hline 1 & 0 & \cdots & 0 & 0 \\ \hline \end{array} \ \boxed{0}\ \textbf{,}\ \text{and } \mathtt{c}''' = \begin{array}{|c|c|c|c|c|}\hline 1 & 0 & \cdots & 0 & 0 \\ \hline 0 & 1 & \cdots & 0 & 0 \\ \hline \end{array} \ \boxed{0}\ \textbf{.}$$

We have $\mathtt{c}' \in \mathbf{comp}_1(\mathtt{a})$, $\mathtt{c}'' \in \mathbf{comp}_1(\mathtt{d})$, and $\mathtt{c}''' \in \mathbf{comp}_1(\mathtt{b})$. As in previous cases, we intend to assign color 1 to the vertex $\mathtt{c}$, but subface-friendliness will require exactly one of the vertices $\mathtt{c}'$ or $\mathtt{c}''$ to also be assigned color 1. We will use vertex $\mathtt{c}''$ for this purpose. It is important to note that $\mathtt{c}''$ has positive depth in $\mathbf{comp}_i(\mathtt{c}'')$ if and only if $i \in \{1, 2\}$; moreover, $\mathbf{comp}_2(\mathtt{c}'')$ is a two-element chain with $\mathtt{c}''$ as its minimal element. Thus, if we assign color 1 to the vertex $\mathtt{c}''$, then this choice will not interfere with the subface-friendliness of the vertex-coloring function that $\mathbf{comp}_1(\mathtt{d})$ inherits when the induction hypothesis is applied.

So, define a vertex-coloring function $\kappa_{m+1,j} : L_{\mathsf{B}}^{DeC}(j, 2m+3-j) \setminus \{\mathbf{max}_{m+1,j}\} \longrightarrow [1, m+1]$ as follows:

$$\kappa_{m+1,j}(\mathtt{x}) = \begin{cases} 1 & \text{if } \mathtt{x} \in \{\mathtt{a}, \mathtt{c}, \mathtt{c}', \mathtt{d}\} \\ \sigma_{m+1}^{-1}\Big((\kappa_{m,j} \circ \phi_{\mathtt{a}})(\mathtt{x})\Big) & \text{if } \mathtt{x} \in \mathbf{comp}_{[2,m+1]}(\mathtt{a}) \text{ and } \mathtt{x} \notin \{\mathtt{a}, \mathtt{c}'\} \\ \sigma_{m+1}^{-1}\Big((\kappa_{m,j-1} \circ \phi_{\mathtt{b}})(\mathtt{x})\Big) & \text{if } \mathtt{x} \in \mathbf{comp}_{[2,m+1]}(\mathtt{b}) \text{ and } \mathtt{x} \neq \mathtt{b} \\ \sigma_{m+1}^{-1}\Big((\kappa_{m,j-1} \circ \phi_{\mathtt{c}})(\mathtt{x})\Big) & \text{if } \mathtt{x} \in \mathbf{comp}_{[2,m+1]}(\mathtt{c}) \text{ and } \mathtt{x} \neq \mathtt{c} \\ \sigma_{m+1}^{-1}\Big((\kappa_{m,j-2} \circ \phi_{\mathtt{d}})(\mathtt{x})\Big) & \text{if } \mathtt{x} \in \mathbf{comp}_{[2,m+1]}(\mathtt{d}) \text{ and } \mathtt{x} \neq \mathtt{d} \end{cases}$$

By our induction hypothesis, $\kappa_{m,j}$, $\kappa_{m,j-1}$, and $\kappa_{m,j-2}$ are subface-friendly. To prove that $\kappa_{m+1,j}$ is subface-friendly, it suffices to check that $\{\mathtt{y} \in \mathbf{comp}_1(\mathtt{x}) \mid \kappa_{m+1,j}(\mathtt{y}) = 1\}$ is a subface of $\mathbf{comp}_1(\mathtt{x})$ for $\mathtt{x} \in \{\mathtt{a}, \mathtt{c}, \mathtt{c}', \mathtt{d}\}$. For $\mathtt{x} \in \{\mathtt{a}, \mathtt{d}\}$, this latter condition is trivial since $\mathbf{comp}_1(\mathtt{x})$ is a two-element chain, $\mathtt{x}$ is its minimal element, and

$\kappa_{m+1,j}(\mathtt{y}) > 1$ if $\mathtt{y} \in \mathbf{comp}_1(\mathtt{x}) \setminus \{\mathtt{x}\}$. Based on the above analysis of the four-element diamond $\mathbf{comp}_1(\mathtt{c})$, we discern that $\kappa_{m+1,j}(\mathtt{y}) > 1$ if $\mathtt{y} \in \mathbf{comp}_1(\mathtt{c}) \setminus \{\mathtt{c}, \mathtt{c}''\}$. So, $\{\mathtt{y} \in \mathbf{comp}_1(\mathtt{x}) \,|\, \kappa_{m+1,j}(\mathtt{y}) = 1\} = \{\mathtt{c}, \mathtt{c}''\}$ is a subface of $\mathbf{comp}_1(\mathtt{x})$ whenever $\mathtt{x} \in \{\mathtt{c}, \mathtt{c}''\}$.

Finally, to complete the induction argument we assume that $j = m+1$. Here, there are exactly four B_m-maximal elements in $L_{\mathsf{B}}^{DeC}(m+1, m+2)$:

$$\mathtt{a} = \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & \cdots & 1 & 1 \\ \hline 0 & 0 & \cdots & 0 & 0 \\ \hline \end{array}\ \boxed{1}\ , \quad \mathtt{b} = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & \cdots & 1 & 1 \\ \hline 0 & 0 & \cdots & 0 & 0 \\ \hline \end{array}\ \boxed{0}\ ,$$

$$\mathtt{c} = \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & \cdots & 1 & 1 \\ \hline 1 & 0 & \cdots & 0 & 0 \\ \hline \end{array}\ \boxed{0}\ , \text{ and } \mathtt{d} = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & \cdots & 1 & 0 \\ \hline 1 & 0 & \cdots & 0 & 0 \\ \hline \end{array}\ \boxed{0}\ .$$

The ellipses on the top row of each of the above tally diagrams indicate a sequence of consecutive 1's beginning in position 3; the bottom row ellipses indicate empty boxes. In the B_m-component of tally diagram $\mathtt{a}$, as we move from the maximal element and towards the minimal element, we can view the 0's as moving counterclockwise from the bottom row of the diagram and into the top row. It is evident, then, that $\mathbf{comp}_{[2,m+1]}(\mathtt{a})$ is isomorphic to $L_{\mathsf{B}}^{DeC}(m, m+1)$. The B_m-components of tally diagrams $\mathtt{b}$ and $\mathtt{c}$ are straightforwardly isomorphic to $L_{\mathsf{B}}^{DeC}(m, m+1)$, and $\mathtt{b} = \mathbf{max}_{m+1,m+1}$ is the unique maximal element of $L_{\mathsf{B}}^{DeC}(m+1, m+2)$. The B_m-component of tally diagram $\mathtt{d}$ is isomorphic to $L_{\mathsf{B}}^{DeC}(m-1, m+2)$. As in the preceding case, $\mathbf{comp}_1(\mathtt{c})$ is a diamond consisting of the following four elements:

$$\mathtt{c} = \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & \cdots & 1 & 1 \\ \hline 1 & 0 & \cdots & 0 & 0 \\ \hline \end{array}\ \boxed{0}\ , \quad \mathtt{c}' = \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & \cdots & 1 & 1 \\ \hline 0 & 1 & \cdots & 0 & 0 \\ \hline \end{array}\ \boxed{0}\ ,$$

$$\mathtt{c}'' = \begin{array}{|c|c|c|c|c|}\hline 1 & 0 & \cdots & 1 & 1 \\ \hline 1 & 0 & \cdots & 0 & 0 \\ \hline \end{array}\ \boxed{0}\ , \text{ and } \mathtt{c}''' = \begin{array}{|c|c|c|c|c|}\hline 1 & 0 & \cdots & 1 & 1 \\ \hline 0 & 1 & \cdots & 0 & 0 \\ \hline \end{array}\ \boxed{0}\ .$$

We have $\mathtt{c}' \in \mathbf{comp}_1(\mathtt{a})$, $\mathtt{c}'' \in \mathbf{comp}_1(\mathtt{d})$, and $\mathtt{c}''' \in \mathbf{comp}_1(\mathtt{b})$. We intend to assign color 1 to vertices $\mathtt{c}$ and $\mathtt{c}''$. Now $\mathtt{c}''$ has positive depth in $\mathbf{comp}_i(\mathtt{c}'')$ if and only if

$i \in \{1,2\}$, and $\mathbf{comp}_2(\mathtt{c}'')$ is a two-element chain with $\mathtt{c}''$ as its minimal element. Thus, if we assign color 1 to the vertex $\mathtt{c}''$, then this choice will not interfere with the subface-friendliness of the vertex-coloring function that $\mathbf{comp}_1(\mathtt{d})$ inherits when the induction hypothesis is applied.

Define a vertex-coloring function $\kappa_{m+1,m+1} : L_{\mathtt{B}}^{DeC}(m+1, m+2) \backslash \{\mathbf{max}_{m+1,m+1}\} \longrightarrow [1, m+1]$ as follows:

$$\kappa_{m+1,m+1}(\mathtt{x}) = \begin{cases} 1 & \text{if } \mathtt{x} \in \{\mathtt{a}, \mathtt{c}, \mathtt{c}', \mathtt{d}\} \\ \sigma_{m+1}^{-1}\Big((\kappa_{m,m} \circ \phi_{\mathtt{a}})(\mathtt{x})\Big) & \text{if } \mathtt{x} \in \mathbf{comp}_{[2,m+1]}(\mathtt{a}) \text{ and } \mathtt{x} \notin \{\mathtt{a}, \mathtt{c}'\} \\ \sigma_{m+1}^{-1}\Big((\kappa_{m,m} \circ \phi_{\mathtt{b}})(\mathtt{x})\Big) & \text{if } \mathtt{x} \in \mathbf{comp}_{[2,m+1]}(\mathtt{b}) \text{ and } \mathtt{x} \neq \mathtt{b} \\ \sigma_{m+1}^{-1}\Big((\kappa_{m,m} \circ \phi_{\mathtt{c}})(\mathtt{x})\Big) & \text{if } \mathtt{x} \in \mathbf{comp}_{[2,m+1]}(\mathtt{c}) \text{ and } \mathtt{x} \neq \mathtt{c} \\ \sigma_{m+1}^{-1}\Big((\kappa_{m,m-1} \circ \phi_{\mathtt{d}})(\mathtt{x})\Big) & \text{if } \mathtt{x} \in \mathbf{comp}_{[2,m+1]}(\mathtt{d}) \text{ and } \mathtt{x} \neq \mathtt{d} \end{cases}$$

By our induction hypothesis, $\kappa_{m,m}$ and $\kappa_{m,m-1}$ are subface-friendly. To prove that $\kappa_{m+1,m+1}$ is subface-friendly, it suffices to check that $\{\mathtt{y} \in \mathbf{comp}_1(\mathtt{x}) \,|\, \kappa_{m+1,m+1}(\mathtt{y}) = 1\}$ is a subface of $\mathbf{comp}_1(\mathtt{x})$ for $\mathtt{x} \in \{\mathtt{a}, \mathtt{c}, \mathtt{c}', \mathtt{d}\}$. For $\mathtt{x} \in \{\mathtt{a}, \mathtt{d}\}$, this latter condition is trivial since $\mathbf{comp}_1(\mathtt{x})$ is a two-element chain, $\mathtt{x}$ is its minimal element, and $\kappa_{m+1,m+1}(\mathtt{y}) > 1$ if $\mathtt{y} \in \mathbf{comp}_1(\mathtt{x}) \setminus \{\mathtt{x}\}$. Based on the above analysis of the four-element diamond $\mathbf{comp}_1(\mathtt{c})$, we see that $\kappa_{m+1,m+1}(\mathtt{y}) > 1$ if $\mathtt{y} \in \mathbf{comp}_1(\mathtt{c}) \setminus \{\mathtt{c}, \mathtt{c}''\}$. So, $\{\mathtt{y} \in \mathbf{comp}_1(\mathtt{x}) \,|\, \kappa_{m+1,m+1}(\mathtt{y}) = 1\} = \{\mathtt{c}, \mathtt{c}''\}$ is a subface of $\mathbf{comp}_1(\mathtt{x})$ whenever $\mathtt{x} \in \{\mathtt{c}, \mathtt{c}''\}$. $\square$

References

[ADLP] L. W. Alverson II, R. G. Donnelly, S. J. Lewis, and R. Pervine, "Constructions of representations of rank two semisimple Lie algebras with distributive lattices," *Electronic J. Combin.* **13** (2006), #R109 (44 pp).

[ADLMPPW] L. W. Alverson II, R. G. Donnelly, S. J. Lewis, M. McClard, R. Pervine, R. A. Proctor, and N. J. Wildberger, "Distributive lattices defined for representations of rank two semisimple Lie algebras," *SIAM J. Discrete Math.* **23** (2009), 527–559.

[Com] L. Comtet, *Advanced Combinatorics*, Reidel, Boston, 1974.

[DeC] C. De Concini, "Symplectic standard tableaux," *Adv. Math.* **34** (1979), 1–27.

[Don1] R. G. Donnelly, "Explicit constructions of representations of semisimple Lie algebras," Ph.D. thesis, University of North Carolina, 1997.

[Don2] R. G. Donnelly, "Symplectic analogs of $L(m, n)$," *J. Comb. Theory Ser. A* **88** (1999), 217–234.

[Don3] R. G. Donnelly, "Explicit constructions of the fundamental representations of the symplectic Lie algebras," *J. Algebra* **233** (2000), 37–64.

[Don4] R. G. Donnelly, "Extremal properties of bases for representations of semisimple Lie algebras," *J. Algebraic Combin.* **17** (2003), 255–282.

[Don5] R. G. Donnelly, "Extremal bases for the adjoint representations of the simple Lie algebras," *Comm. Algebra*, **34** (2006), 3705–3742.

[Don6] R. G. Donnelly, "Eriksson's numbers game and finite Coxeter groups," *European J. Combin.* **29** (2008), 1764–1781.

[Don7] R. G. Donnelly, "Poset models for Weyl group analogs of symmetric functions and Schur functions," unpublished research manuscript, 2018.

[DDDS] R. G. Donnelly, E. A. Donovan, M. W. Dunkum, and T. A. Schroeder, "Move-mini-mizing games and diamond-colored modular and distributive lattices (unabridged)," `arXiv`.

[DDS] R. G. Donnelly, E. Donovan, and T. A. Schroeder, "Domino games and lattice models for fundmental representations and elementary Weyl group symmetric functions: The type A and C cases," in preparation.

[FH] W. J. Fulton and J. Harris, *Representation Theory: A First Course*, Graduate Texts in Mathematics vol. 129, Springer-Verlag, New York, 1991.

[Gil] M. R. Gilliland, "Distributive Lattices and Weyl Characters of Exotic Type F_4," Master's thesis, Murray State University, 2008. Available at:
`http://campus.murraystate.edu/academic/faculty/rdonnelly/Research/MGThesis.pdf`

[HL] P. Hersh and C. Lenart, "Combinatorial constructions of weight bases: the Gelfand-Tsetlin basis," *Electronic J. Combin.* **17** (2010), #R33 (14 pp).

[Hum] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics vol. 8, Springer-Verlag, New York, 1972.

[Kac] V. G. Kac, *Infinite-dimensional Lie Algebras*, 3rd edition, Cambridge University Press, Cambridge, 1990.

[KN] M. Kashiwara and T. Nakashima, "Crystal graphs for representations of the q-analogue of classical Lie algebras," *J. Algebra* **165** (1994), 295–345.

[Kum] S. Kumar, *Kac–Moody Groups, Their Flag Varieties and Representation Theory*, Birkhäuser Boston Inc, Boston, MA, 2002.

[O] K. M. O'Hara, "Unimodality of Gaussian coefficients: a combinatorial proof," *J. Combin. Theory Ser. A* **53** (1990), 29–52.

[Proc1] R. A. Proctor, "Solution of two difficult combinatorial problems with linear algebra," *Amer. Math. Monthly* **89** (1982), 721–734.

[Proc2] R. A. Proctor, "Bruhat lattices, plane partition generating functions, and minuscule representations," *European J. Combin.* **5** (1984), 331–350.

[She] J. T. Sheats, "A symplectic jeu de taquin bijection between the tableaux of King and of De Concini," *Trans. Amer. Math. Soc.* **351** (1999), 3569–3607.

[Sag] B. Sagan, *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, 2_{nd} ed., Springer, New York, 2001.

[Stan1] R. P. Stanley, "Differential posets," *Journal of the American Mathematical Society* **1** (1988), 919–961.

[Stan2] R. P. Stanley, *Enumerative Combinatorics, Vol. 2*, Cambridge University Press, 1999.

[Zeil] D. Zeilberger, "Kathy O'Hara's constructive proof of the unimodality of the Gaussian polynomials," *Amer. Math. Monthly* **96** (1989), 590–602.