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Quantum Dimension Polynomials: A Networked-Numbers Game Approach

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Quantum Dimension Polynomials: A Networked-Numbers Game Approach

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Nicholas Gaubatz

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Abstract

The Networked-Numbers Game–a mathematical "game" played on a simple graph–is incredibly accessible and yet surprisingly rich in content. The Game is known to contain deep connections to the finite-dimensional simple Lie algebras over the complex numbers. On the other hand, Quantum Dimension Polynomials (QDPs)–enumerative expressions traditionally understood through root systems–corresponding to the above Lie algebras are complicated to derive and often inaccessible to undergraduates. In this thesis, the Networked-Numbers Game is defined and some known properties are presented. Next, the significance of the QDPs as a method to count combinatorially interesting structures is relayed. Ultimately, a novel closed-form expression of the type D_n QDPs and novel derivations of the QDPs of types A_n, B_n, C_n, and D_n are provided using an inductive proof through the Networked-Numbers Game. This provides a combinatorial avenue of approach to a topic traditionally only attainable through Lie theory.

Contents

Chapter 1

Introduction

The purpose of this text is twofold. First, it serves as an entry point into the Networked-Numbers Game (NG) and some related mathematical objects in a way that is approachable by a typical undergraduate mathematics student. Second, it develops in some detail some NG-related aspects of so-called 'Quantum Dimension Polynomials.' In particular, it offers closed-form expressions for QDPs in types A_n, B_n, C_n, and D_n and shows how these expressions can be obtained from NG play. It also provides a few applications to an extension to the n-choose-k function and crystal graphs.

As introduced in Chapter 2, the NG is developed first naively and then formally through an algebraic perspective. The chapter ends with an important classification theorem relating the NG to finite-dimensional simple Lie algebras. The NG provides a nice entry point into our combinatorial perspective of QDPs, with derivations/proofs of QDPs in types A_n, B_n, C_n, and D_n

comprising most of the content of Chapter 3 and Appendix A. These proofs consist mostly of sequences of NG states with generalized populations. Chapter 4 briefly touches on two combinatorial applications of QDPs—an identity relating the QDPs in type A_n to q-binomial coefficients, and q-enumeration of k-element subsets of an $(n+1)$-element set, respectively.

Lastly, note that most of the enumerative and order-theoretic constructs considered in this paper follow the conventions of [Stan1] and [Stan2].

Chapter 2

The Networked-Numbers Game

Before we start discussing Quantum Dimension Polynomials and their uses, it is important to understand what the Networked-Numbers Game (NG) is. The NG is a “game” played on a finite simple graph whose edges are labeled by values from an associated matrix. The game itself is often attributed to Mozes [Moz], who was inspired by a Math Olympiad problem. Erikson [Erik1], [Erik2], and [Erik3] seems to have conceived of the game in the level of generality we consider here. However, our setup will more closely follow the more recent work [Don1].

In the following, we first present an “informal introduction,” then a precise definition of the NG, and lastly an important classification theorem.

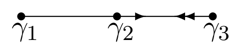
2.1 An informal introduction to the Networked-Numbers Game

To set up the NG, first choose a finite simple graph Γ with n nodes (the nomenclature of 'nodes' is more common than 'vertices' in references that consider the NG). To each node $\gamma_i \in V(\Gamma)$ assign a *population*—a nonnegative real number λ_i—at least one of which is nonzero. Additionally, to each edge in Γ between nodes γ_i and γ_j assign two negative integers a_{ij} and a_{ji}. Their absolute values, seen shortly in an example, will act as *amplifiers*, and a value of $|a_{ij}| > 1$ is often denoted on Γ by drawing $|a_{ij}|$ arrows on the edge between γ_i and γ_j (this can be seen in Example 2.1.1).

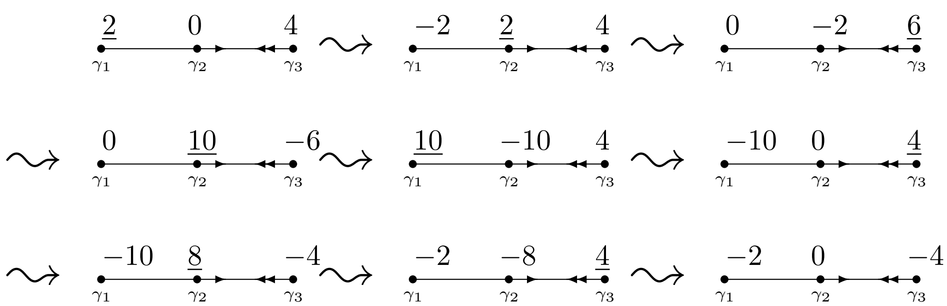
The goal of the game is to successively "fire" nodes with positive populations in such a way that eventually each population is nonpositive; that is, for each i such that $1 \leq i \leq |V(\Gamma)|$, $\lambda_i \leq 0$. Examples of such setups are given in Examples 2.1.1 and 2.1.2.

To *fire* a given node γ_i with a positive population λ_i in a given *state*, simply update each population λ_j at node γ_j, $1 \leq j \leq n$ and $j \neq i$, as follows: if γ_j is not adjacent to γ_i, keep λ_j constant; otherwise, $\lambda_j \mapsto \lambda_j + |a_{ij}|\lambda_i$. This can be understood as taking λ_i, 'amplifying' it by $|a_{ij}|$, and adding it to λ_j. Note that the quantity $\lambda_j + |a_{ij}|\lambda_i$ is the same as $\lambda_j - a_{ij}\lambda_i$ since we have the convention that $a_{ij} < 0$. From here on, we'll use the latter expression, which is more common in the literature. NG play stipulates that a firing move is only 'legal' when the population at the to-be-fired node is positive. Lastly, update λ_i by changing its sign. We repeat this legal node-firing process iteratively, producing a (possibly infinite) sequence of NG states.

Example 2.1.1. Consider the graph denoted by $\Gamma = C_3$ (for reasons explained later):

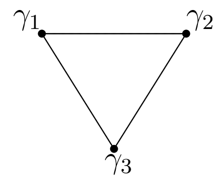


Note that in this case, the arrows on the edge between γ_2 and γ_3 represent $a_{23} = -1$ and $a_{32} = -2$. Let us assign initial population $(2, 0, 4)$ to $(\gamma_1, \gamma_2, \gamma_3)$ and play the NG as follows. To keep things consistent, whenever we are faced with multiple firing choices, we will choose the leftmost node to fire. This choice produces the following state sequence:

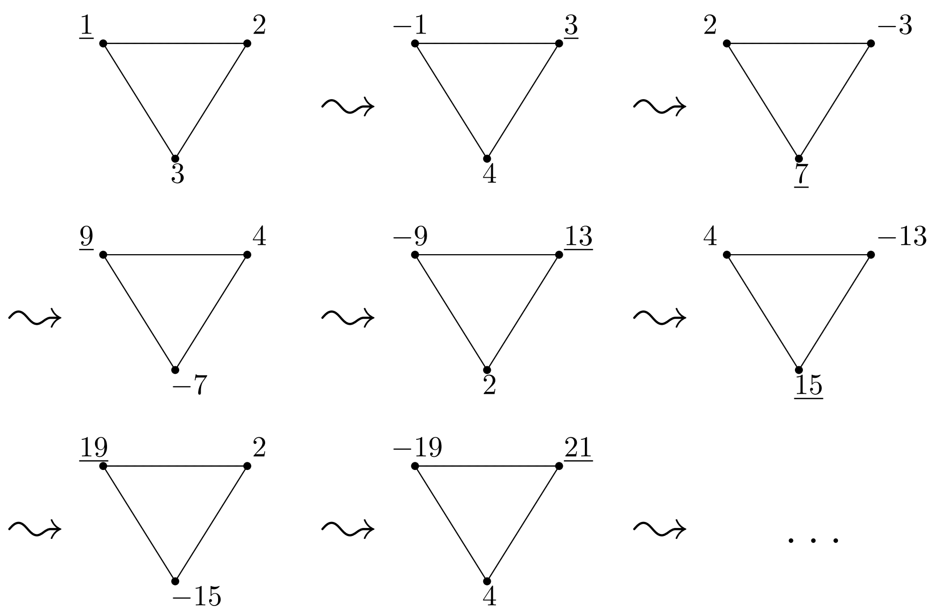


Note that the game does indeed terminate with this choice of $\Gamma = C_3$ and initial population $(\lambda_1, \lambda_2, \lambda_3) = (2, 0, 4)$. The next example will show a game that does not terminate.

Example 2.1.2. Now let our graph Γ be the standard 3-cycle with all amplitudes $a_{ij} = -1$ as follows:



Similarly as above, let us assign initial population $(\lambda_1, \lambda_2, \lambda_3) = (1, 2, 3)$. We will choose firing sequence $(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \ldots)$ to produce the following state sequence:



This game appears to never terminate. This can be seen by considering the sum of populations throughout the state sequence. Upon firing node γ_i, note that we subtract λ_i twice from the sum (at node γ_i) but also add λ_i twice to the sum, once at each of the other nodes. Thus, the sum stays constant no matter which node is fired; but, in order to reach a terminal state, the sum would need to eventually be negative.

This indeed shows that the NG played on this 3-cycle never terminates, no matter the starting population or firing sequence. In fact, "most" graphs have no terminal games, as demonstrated in a special classification theorem in Section 2.3. But, before we can state this classification, we must more formally define the Networked-Numbers Game.

2.2 The formal Networked-Numbers Game

To this end, let I be a finite set, whose elements can be thought of as indices or colors (i.e., I is a *coloring set*). Next, let Γ be a simple graph—i.e., Γ has no loops or multiple edges—whose vertices or nodes are $V(\Gamma) = \{\gamma_i\}_{i \in I}$ and whose edges are $E(\Gamma)$, which consists of two-element subsets of $V(\Gamma)$, each corresponding to an edge and its unique endpoints. Lastly, let $A = (a_{ij})_{i,j \in I}$ be an $I \times I$ integer matrix satisfying

$$a_{ij} \begin{cases} = 2, \text{ if } i = j \\ = 0, \text{ if } \{i, j\} \notin E(\Gamma) \\ < 0, \text{ if } \{i, j\} \in E(\Gamma). \end{cases}$$

This matrix is used to label the edges of Γ, as in section 2.1.

We call $\mathscr{G} = (\Gamma, A)$ a *game graph* or *NG graph*. We also call $\Gamma = \Gamma(\mathscr{G})$

its *playground graph* and $A = A(\mathscr{G})$ its *amplitude matrix*. To emphasize the role of I, we sometimes write Γ_I or $A_{I\times I}$.

A playthrough of the NG consists of a sequence of *states* and *node firings*; the node firings are used to determine the successor of a given state.

To play the NG, the player chooses a game graph $\mathscr{G} = (\Gamma_I, A_{I\times I})$ and an *initial state*, an I-tuple of nonnegative integers $\lambda := \{\lambda_i\}_{i\in I}$, at least one of which is nonzero. From this point on, we denote by *state* any integer I-tuple $\mu = (\mu_i)_{i\in I}$. Each μ_i located at position i is called the *population* at node γ_i. Given a state μ, the only move a player can make is to choose a node γ_i whose population μ_i is positive and *fire* it. This firing transforms the current state μ into a new state ν according to the rule

$$\nu_j = \mu_j - a_{ij}\mu_i,$$

for all $j \in I$, where $a_{ij} \in A$. Note that this is the same process as in Section 2.1, since $a_{ii} = 2$ for all $i \in I$ (equivalent to saying $\nu_i = -\mu_i$) and $a_{ij} = 0$ if γ_i is not adjacent to γ_j. Additionally, the value a_{ij} from the amplitude matrix A does indeed, in a sense, 'amplify' μ_i before we subtract it from μ_j.

Thus, gameplay begins by starting with an initial state λ and applying legal node-firings to resulting states. Gameplay ends when the player obtains a *terminal state*, in which all populations are nonpositive.

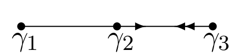
Algebraically, a state λ can be viewed as a vector $\sum\limits_{i\in I} \lambda_i\omega_i$ in the $\mathbb{Z}$-module Λ freely generated by the set $\Omega = \{\omega_i\}_{i\in I}$. Elements of Ω are *fundamental* *weights* (or *fundamental positions*), and elements of Λ are *weights* or *game positions*. A weight or position $\lambda = \sum_{i \in I} \lambda_i \omega_i$ is *dominant* (*strongly dominant*) if $\lambda_i \geq 0$ ($\lambda_i > 0$) for all $i \in I$.

Let $S_i : \Lambda \to \Lambda$ be the $\mathbb{Z}$-linear transformation given by the i-th NG firing move (but without regard to legality):

$$\begin{aligned} S_i(\lambda) &= \lambda - \lambda_i \alpha_i \\ &= \sum_{j \in I} (\lambda_j - \lambda_i a_{ij}) \omega_j, \end{aligned}$$

where α_i is the i-th row vector of A, so $\alpha_i = \sum_{j \in I} a_{ij} \omega_j$. Since $S_i^2 = \varepsilon$ (the identity $\mathbb{Z}$-linear transformation $\Lambda \to \Lambda$) in the group $\mathrm{Aut}(\Lambda)$ of invertible $\mathbb{Z}$-linear transformations on Λ, we can consider the Weyl group $W \leq \mathrm{Aut}(\Lambda)$ generated by the S_i's. Now, it is known that $W \cong \langle s_i | (s_i s_j)^{m_{ij}} = \varepsilon \rangle$, where m_{ij} is the unique positive integer for which $a_{ij} a_{ji} = 4 \cos^2(\frac{\pi}{m_{ij}})$. Also, the 'parity' function $\mathrm{sgn} : W \to \{\pm 1\}$ given by $\mathrm{sgn}(s_{i_1} \cdots s_{i_k}) := (-1)^k$ is a well-defined group homomorphism [Hum].

Example 2.2.1. Consider W for $\Gamma = C_3$ as follows,



,

where we have that

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{bmatrix}.$$

In this case, calculating the m_{ij}'s gives that

$$\begin{aligned}
a_{12}a_{21} = (-1)(-1) = 1 = 4\cos^2\left(\frac{\pi}{m_{12}}\right) &\implies m_{12} = 3, \\
a_{13}a_{31} = (0)(0) = 0 = 4\cos^2\left(\frac{\pi}{m_{13}}\right) &\implies m_{13} = 2, \text{ and} \\
a_{23}a_{32} = (-1)(-2) = 2 = 4\cos^2\left(\frac{\pi}{m_{23}}\right) &\implies m_{23} = 4.
\end{aligned}$$

Therefore,

$$\begin{aligned}
W &\cong \langle s_i | (s_i s_j)^{m_{ij}} = \varepsilon \rangle \\
&\cong \langle s_1, s_2, s_3 | s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_1 s_3)^2 = (s_2 s_3)^4 = \varepsilon \rangle,
\end{aligned}$$

a group that is known to have order 48.

Now that we have defined the NG, the natural question to ask at this point is what choices of NG graphs $\mathscr{G} = (\Gamma, A)$ and initial states eventually produce terminal states; i.e., finite gameplay?

2.3 The *La Florado Klasado* classification

In order to answer this finiteness question, let us provide a few definitions.

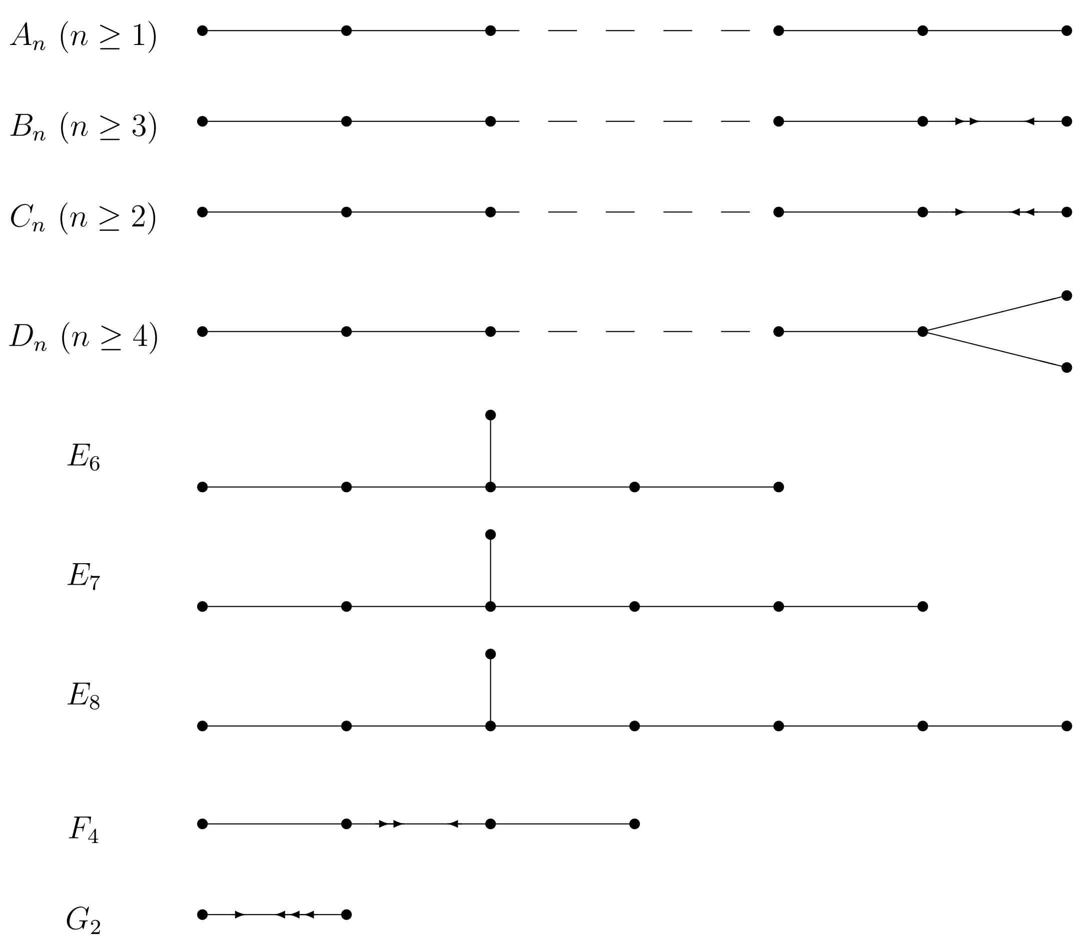
First, a NG state μ on graph $\mathscr{G} = (\Gamma_I, A_{I\times I})$ is *nonzero* if there exists $i \in I$ such that $\mu_i \neq 0$. Next, a state is *dominant* if $\mu_i \geq 0$ for all $i \in I$. The graphs that concern us presently are called *integer game graphs* (ING graphs), in which each amplitude $a_{ij} \in A$ is an integer.

Definition. A *game-gratifying graph* is a connected ING graph $\mathscr{G} = (\Gamma, A)$ that has a nonzero dominant initial state from which a terminal state can be reached.

We are now ready to state a classification of game-gratifying graphs. This classification forms one part of a multifaceted theorem sometimes referred to as *La Florado Klasado* (often denoted LFK), as in section 9 of [Don1]. This theorem refers to Figure 2.1 on the next page. The most famous instance of an LFK equivalence is the classification of the finite-dimensional simple Lie algebras over $\mathbb{C}$ accomplished by W. Killing and E. Cartan in the late 1800s; see [Col] for a compelling account of the discovery of the latter classification.

Theorem 1 (La Florado Klasado)**.** *Suppose $\mathscr{G}$ is a connected integral NG graph. Then $\mathscr{G}$ is game-gratifying if and only if $\mathscr{G}$ is a Coxeter-Dynkin flower; i.e., one of the NG graphs of Figure 2.1.*

Figure 2.1: The Coxeter-Dynkin flowers of LFK



Chapter 3

The Quantum Dimension Polynomial Identity

3.1 A brief introduction to QDPs through Weyl symmetric function theory

Now that we have defined the Networked-Numbers Game on an integral NG graph, it is time to present the main result of this work: a description of Quantum Dimension Polynomials (QDPs). This Section 3.1 lays out many definitions based on terminology introduced in Section 2.2 and is not necessary for an initial understanding of QDPs, but it serves as a brief reference for the context of QDPs in Weyl symmetric function theory. Throughout this section, assume $\mathscr{G}$ is a Coxeter-Dynkin flower.

Let $\{z_i\}_{i \in I}$ be a set of indeterminates. For any $\mu = \sum_{i \in I} \mu_i \omega_i$, let z^μ be the Laurent monomial $\prod_{i\in I} z_i^{\mu_i}$. Note that state vectors are integer I-tuples, so the monomial exponents here are integers. Let $\mathscr{L}(\mathscr{G})$ be the $\mathbb{Z}$-algebra of Laurent polynomials in the variables $\{z_i\}_{i\in I}$ with coefficients from $\mathbb{Z}$. The Weyl group W acts on $\mathscr{L}(\mathscr{G})$ by the rule $s_i \cdot z^{\mu} := z^{s_i\cdot\mu}$, extending $\mathbb{Z}$-linearly to all of W in the obvious way.

Say $\chi \in \mathscr{L}(\mathscr{G})$ is W-invariant, or a Weyl-symmetric function, if $\sigma \cdot \chi = \chi$ for all $\sigma \in W$. Let $\mathscr{L}(\mathscr{G})^W$ be the $\mathbb{Z}$-subalgebra of Weyl symmetric functions.

For each strongly dominant $\mu \in \Lambda$, define an 'alternant' $\mathcal{A}_\mu$ by

$$\mathcal{A}_\mu := \sum_{\sigma\in W} \mathrm{sgn}(\sigma) z^{\sigma\cdot\mu}.$$

Let $\varrho := \sum \omega_i$ be the 'smallest' strongly dominant weight. A fundamental theorem in the theory of Weyl symmetric functions is that, for each dominant λ, there is a unique solution $\chi \in \mathscr{L}(\mathscr{G})^W$ such that

$$\mathcal{A}_\varrho \cdot \chi = \mathcal{A}_{\lambda+\varrho}.$$

This unique Weyl symmetric function is denoted χ_λ and called a Weyl bialternant. By specializing each z_i at a certain power of q, we get a q-polynomial called the *quantum dimension polynomial*, denoted $qdim_\lambda^{\mathscr{G}}$.

A quotient-of-products expression for $qdim_\lambda^{\mathscr{G}}$ seems first to have been given in [Jac] where the product is taken over the so-called 'positive roots' of the 'root system' associated with $\mathscr{G}$. In [Don2], a connection is made between positive roots and states of NG play from a generic strongly dominant weight. This connection is the basis for the following result, presented as Theorem 10.6.2 in [Don1]:

Theorem 2 (Theorem NG). *Given a dominant weight λ, NG play from initial position $\lambda + \varrho$ must terminate on our given Coxeter-Dynkin flower $\mathscr{G}$. Let $(\gamma_{i_1}, \ldots, \gamma_{i_l})$ be such a terminating game, and let $c_k(\lambda + \varrho)$ be the positive number at node γ_{i_k} just before this node is fired in our given game sequence. Then the multiset of numbers $\{c_k(\lambda + \varrho)\}_{k=1}^{l}$ does not depend on the choice of game sequence, and*

$$qdim_{\lambda}^{\mathscr{G}} = \prod_{k=1}^{l} \frac{1 - q^{c_k(\lambda+\rho)}}{1 - q^{c_k(\varrho)}}.$$

We will employ Theorem NG in the next section in our derivation of certain quotient-of-product expressions for $qdim_{\lambda}^{\mathscr{G}}$ when $\mathscr{G} \in \{A_n, B_n, C_n, D_n\}$.

Given any polynomial in q with positive integer coefficients, it is reasonable to ask what the polynomial might enumerate at $q = 1$, especially if the polynomial has a nice product expression. One might further ask what the positive integer coefficients enumerate.

For quantum dimension polynomials, there is a nice combinatorial answer to these questions: $qdim_{\lambda}^{\mathscr{G}}$ is the rank-generating function for an edge-colored and ranked poset called a 'crystal graph' which we denote here by $R(\lambda)$. A distinguishing feature of crystal graphs is that they are 'fibrous' in that all single-color components of any $R(\lambda)$ are chains. The notion of a crystal graph was developed by Kashiwara in [Kash1] and [Kash2] as a distilling of information about so-called crystal bases for representations of quantum groups. The precise definition of a crystal graph is beyond the scope of this thesis, but we can describe how one can, in practice, construct them.

To begin, one constructs a 'seed graph' from NG play, and then one grows all other crystal graphs from this seed. By playing the Numbers Game with the rows α_i of A as initial positions, we can construct a certain $\mathscr{G}$-structured fibrous poset $A(\mathscr{G})$ called the adjoint crystal graph. This graph is special in that, for any dominant weight λ, there is a 'crystal power' $A(\mathscr{G})^{\otimes m}$ of $A(\mathscr{G})$ (of Section 5 of [Don3]) which has a ($\mathscr{G}$-structured and fibrous) connected component $R(\lambda)$ whose unique maximal element has weight λ and

$$WGF(R(\lambda); z) = \chi_{\lambda}^{\mathscr{G}}, \text{ and}$$
$$RGF(R(\lambda); q) = qdim_{\lambda}^{\mathscr{G}}.$$

It is in this sense that Quantum Dimension Polynomials can be seen as rank-generating functions.

3.2 Our QDP Theorem, and proofs for types A_n and D_n

For a given integral NG graph $\mathscr{G}$ and weight λ, the corresponding QDP is represented by $qdim_{\lambda}^{\mathscr{G}}$. These enumerative expressions are already defined in the literature, but in the proof of Theorem 3 we provide an NG-based approach for deriving these polynomials. For the sake of brevity, we only include proofs of the A_n and D_n cases here, while the B_n and C_n arguments can be found in Appendix A.

Two quick notes about notation: for $i, j \in \mathbb{Z}^+$, $i \leq j$, let

$$\lambda_i^j := \sum_{m=i}^{j} \lambda_m = \lambda_i + \lambda_{i+1} + \cdots + \lambda_j.$$

For $i > j$, let $\lambda_i^j := 0$. Additionally, for $k \in \mathbb{Z}^+$, define the q-integer to be

$$[k]_q := 1 + q + \cdots + q^{k-1} = \frac{1-q^k}{1-q}.$$

We use the rational function form as a simplification of the polynomial and regard the discontinuity at $q = 1$ as removable.

Theorem 3 (The Quantum Dimension Polynomial Identity)**.** *Consider the integral NG graphs of type A_n, B_n, C_n, and D_n named in Theorem 1. Let $\lambda = (\lambda_i)_{i=1}^n$ be a nonnegative weight. Then*

$$qdim_\lambda^{A_n} = \prod_{i=1}^{n} \prod_{j=i}^{n} \frac{[\lambda_i^j + j + 1 - i]_q}{[j+1-i]_q},$$

$$qdim_\lambda^{B_n} = \prod_{i=1}^{n-1} \prod_{j=i}^{n-1} \frac{[\lambda_i^j + j + 1 - i]_q}{[j+1-i]_q} \prod_{i=1}^{n} \prod_{j=i}^{n} \frac{[\lambda_i^n + \lambda_j^{n-1} + 2n + 1 - i - j]_q}{[2n+1-i-j]_q},$$

$$qdim_\lambda^{C_n} = \prod_{i=1}^{n-1} \prod_{j=i}^{n-1} \frac{[\lambda_i^j + j + 1 - i]_q}{[j+1-i]_q} \prod_{i=1}^{n} \prod_{j=i+1}^{n+1} \frac{[\lambda_i^n + \lambda_j^n + 2n + 2 - i - j]_q}{[2n+2-i-j]_q}, \text{ and}$$

$$qdim_\lambda^{D_n} = \prod_{i=1}^{n-1} \prod_{j=i}^{n-1} \frac{[\lambda_i^j + j + 1 - i]_q}{[j+1-i]_q} \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} \frac{[\lambda_i^{n-2} + \lambda_j^n + 2n - i - j]_q}{[2n-i-j]_q}.$$

Furthermore, the above polynomials can be obtained by playing through the corresponding NG graph with initial weight $(\lambda_i + 1)_{i=1}^{n}$*, recording each fired population* μ_i *as* $[\mu_i]_q$*, and multiplying each q-integer together.*

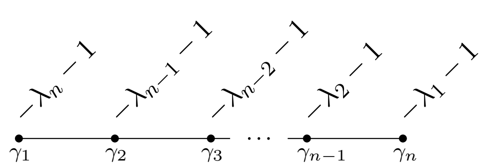
Proof. **(Case A_n):**

We will use induction to establish the following three claims for any $n \in \mathbb{Z}^+$:

Using firing sequence $(\gamma_1; \gamma_2, \gamma_1; \gamma_3, \gamma_2, \gamma_1; \ldots; \gamma_n, \gamma_{n-1}, \ldots, \gamma_1)$ on A_n, starting with initial weight $\lambda = (\lambda_1 + 1, \ldots, \lambda_n + 1) = \sum_{i=1}^{n} (\lambda_i + 1)\omega_i$, we get

1. The above $qdim_\lambda^{A_n}$ q-polynomial equality holds by taking the fired weights,

2. The sequence of NG states ends with terminal state



, and

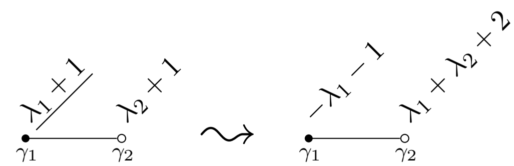
3. At an attached "ghost node" γ_{n+1} with initial weight $\lambda_{n+1} + 1$ to node γ_n, the terminal weight of γ_{n+1} is

$$\lambda_1^{n+1} + (n+1) = \lambda_1 + \cdots + \lambda_{n+1} + (n+1) = \sum_{i=1}^{n+1} (\lambda_i + 1).$$

   This claim is essential to the induction step of the proof.

(Case $n = 1$):

Playing the NG on A_1 using firing sequence (γ_1)–keeping in mind the "ghost node" γ_2, we get



Thus,

1. $\dfrac{[\lambda_1 + 1]_q}{[1]_q} = \dfrac{[\lambda_1^1 + 1 + 1 - 1]_q}{[1 + 1 - 1]_q} = \displaystyle\prod_{i=1}^{1}\prod_{j=i}^{1} \frac{[\lambda_i^j + j + 1 - i]_q}{[j + 1 - i]_q}$,

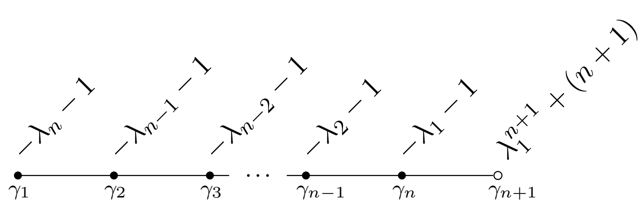
2. The terminal state is the same as we desire, and

3. The terminal weight of the "ghost node" γ_2 is $\lambda_1 + \lambda_2 + 2 = \sum_{i=1}^{2}(\lambda_i + 1)$.

Therefore, our proposition holds for $n = 1$.

(Case $n \Rightarrow n + 1$):

Now, suppose that our three hypotheses hold for some $n \in \mathbb{Z}^+$. That is, the result for our gameplay so far on A_n is the following picture:

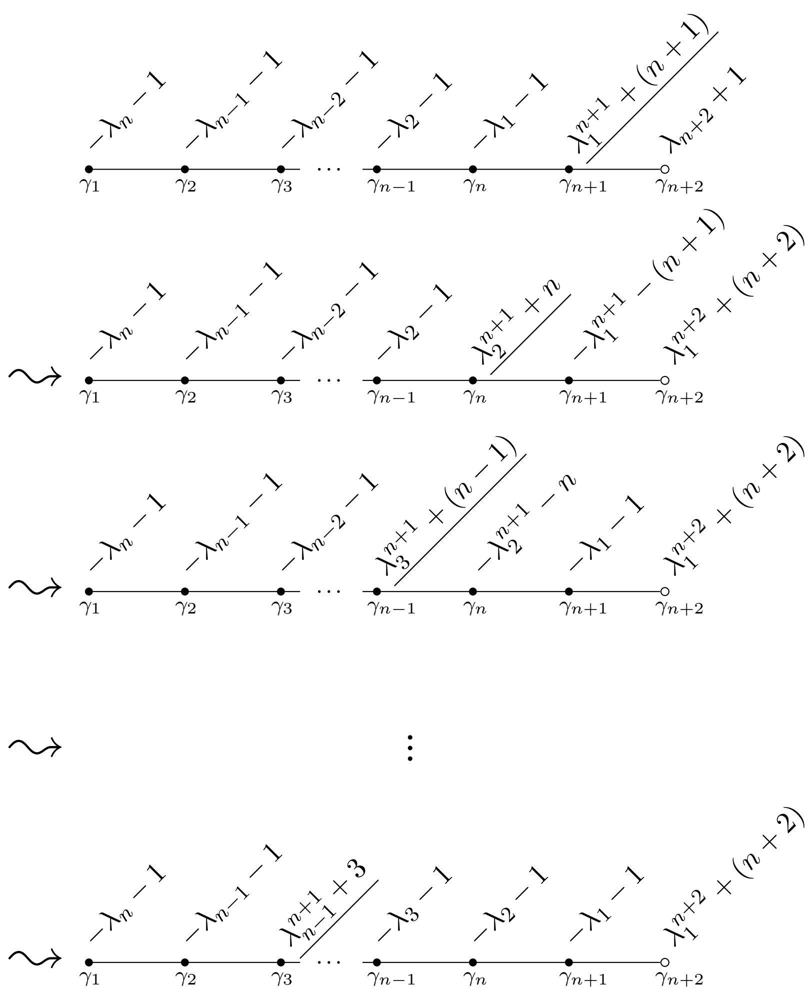


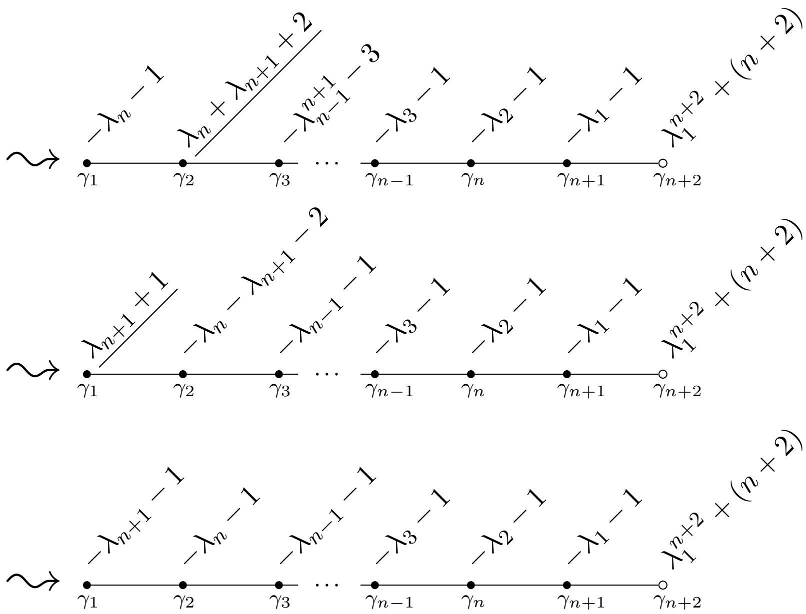
.

We also get that

$$qdim_{\lambda}^{A_n} = \prod_{i=1}^{n}\prod_{j=i}^{n}\frac{[\lambda_i^j + j + 1 - i]_q}{[j+1-i]_q}.$$

Now continue gameplay on A_{n+1} by adding on a new "ghost node" $\gamma_{(n+1)+1} = \gamma_{n+2}$ with population $\lambda_{n+2}+1$ and appending to the current game the firing sequence $(\gamma_{n+1}, \gamma_n, \ldots, \gamma_1)$. We get the following game sequence:



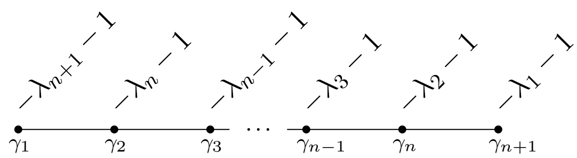


Then by applying Theorem NG, we get

1.

$$
\begin{aligned}
qdim_\lambda^{A_{n+1}} &= qdim_\lambda^{A_n} \prod_{k=1}^{n+1} \frac{[\lambda_k^{n+1} + (n+1) + 1 - k]_q}{[(n+1) + 1 - k]_q} \\
&= \prod_{i=1}^{n} \prod_{j=1}^{n} \frac{[\lambda_i^j + j + 1 - i]_q}{[j+1-i]_q} \prod_{k=1}^{n+1} \frac{[\lambda_k^{n+1} + (n+1) + 1 - k]_q}{[(n+1) + 1 - k]_q} \\
&= \prod_{i=1}^{n+1} \prod_{j=1}^{n+1} \frac{[\lambda_i^j + j + 1 - i]_q}{[j+1-i]_q}.
\end{aligned}
$$

2. Terminal state



3. The weight of the new "ghost node" $\gamma_{(n+1)+1}$ is

$$\begin{aligned}\lambda_1^{n+2}+(n+2) &= \lambda_1+\cdots+\lambda_{n+1}+\lambda_{(n+1)+1}+((n+1)+1)\\ &= \sum_{i=1}^{(n+1)+1}(\lambda_i+1).\end{aligned}$$

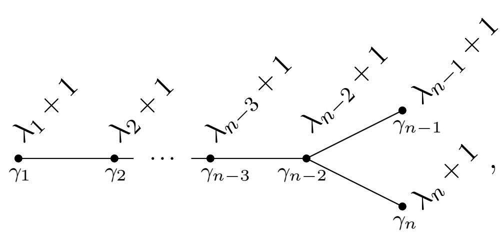
Therefore, the proposition holds for all $n \in \mathbb{Z}^+$.

(Case B_n): See Appendix A.

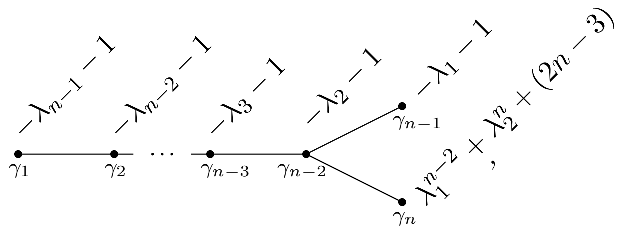
(Case C_n): See Appendix A.

(Case D_n):

Note that $\prod_{i=1}^{n-1}\prod_{j=i}^{n-1}\frac{[\lambda_i^j+j+1-i]_q}{[j+1-i]_q} = qdim_\lambda^{A_{n-1}}$. This falls in line with the fact that when viewing D_n as follows,

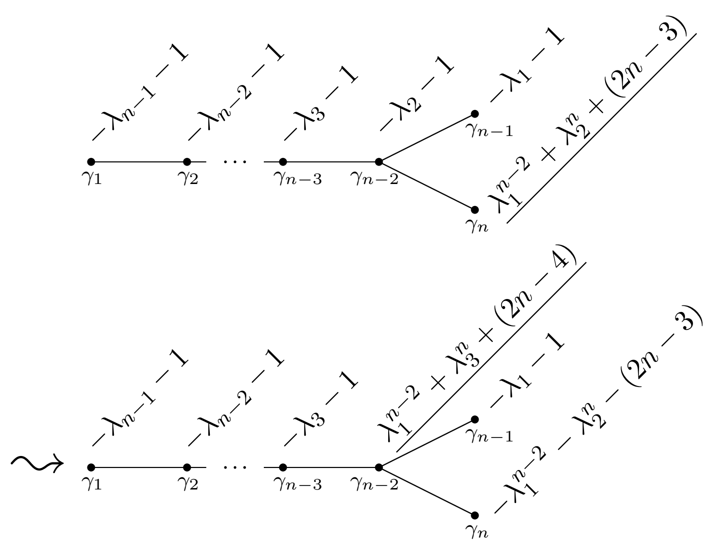


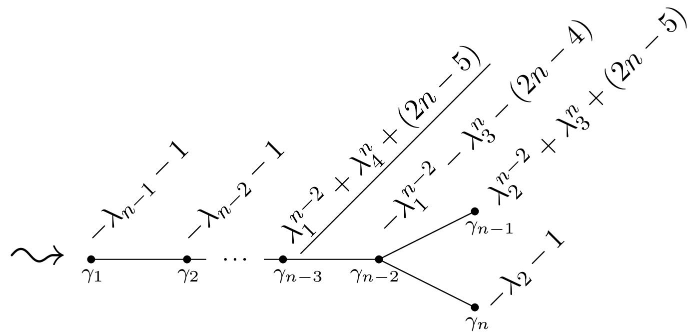
removing γ_n leaves a copy of A_{n-1}. In fact, starting with firing sequence $(\gamma_1;\gamma_2,\gamma_1;\ldots;\gamma_{n-2},\gamma_{n-3},\ldots,\gamma_2,\gamma_1;\gamma_{n-1},\gamma_{n-2},\gamma_{n-3},\ldots,\gamma_2,\gamma_1)$ produces



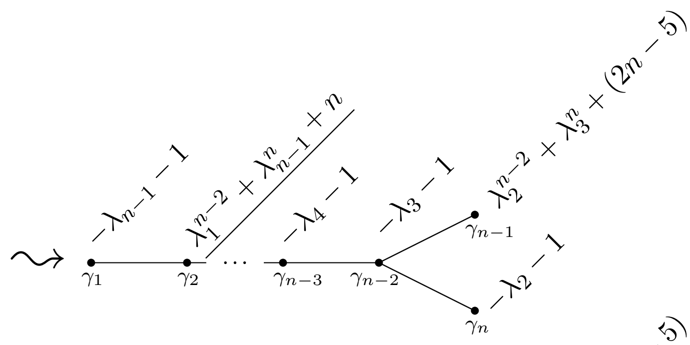
since the only two times γ_{n-2} is fired it first has population $\lambda_1^{n-2}+(n-2)$ and then $\lambda_1^{n-1}+(n-1)$, which are each added to (λ_n+1) at γ_n to give population $\lambda_1^{n-2}+\lambda_2^n+(2n-3)$.

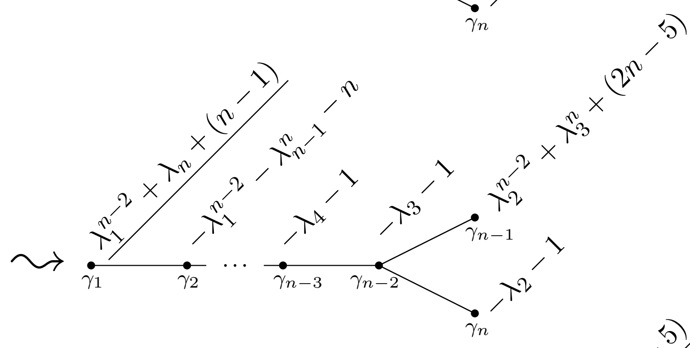
All that is left at this point is the second double product of the proposed $qdim_\lambda^{D_n}$. The firing sequence we choose will depend on whether n is even or odd. For even n, continuing gameplay using firing sequence $(\gamma_n, \gamma_{n-2}, \gamma_{n-3}, \ldots, \gamma_2, \gamma_1; \gamma_{n-1}, \gamma_{n-2}, \gamma_{n-3}, \ldots, \gamma_2; \ldots; \gamma_n, \gamma_{n-2}, \gamma_{n-3}; \gamma_{n-1}, \gamma_{n-2}; \gamma_n)$ gives us the following:

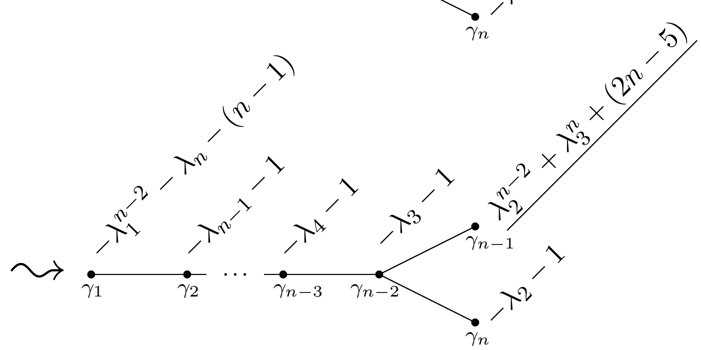




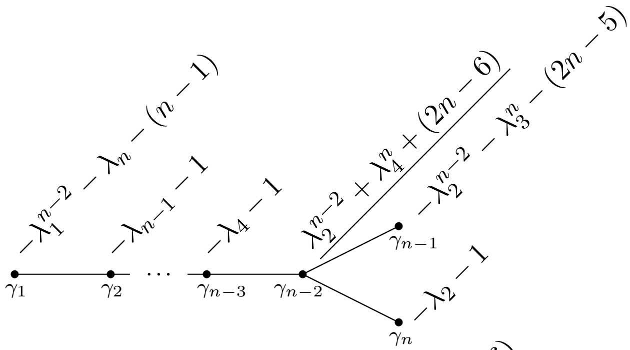
$\rightsquigarrow \qquad \vdots$



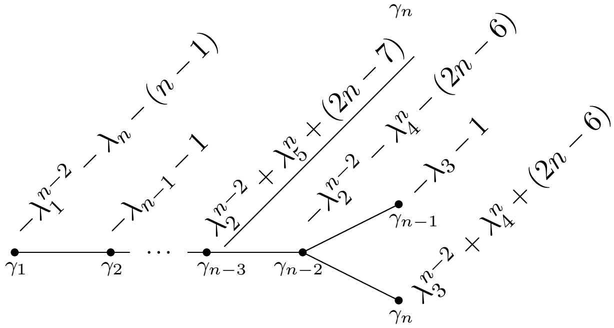




$\rightsquigarrow$

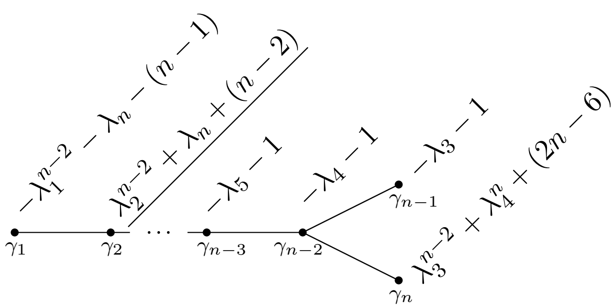


$\rightsquigarrow$



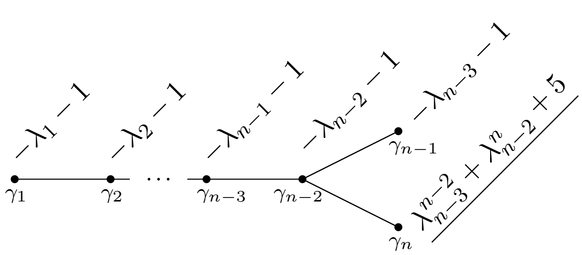
$\rightsquigarrow$ $\vdots$

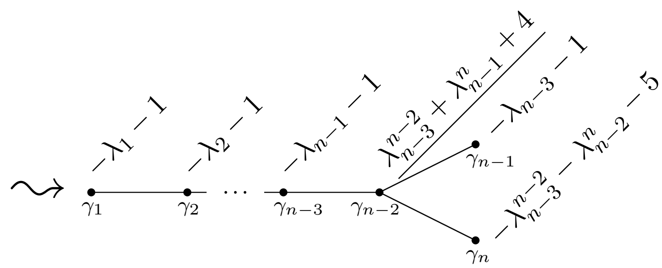
$\rightsquigarrow$

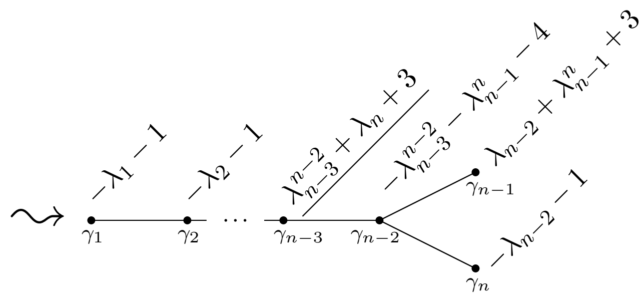


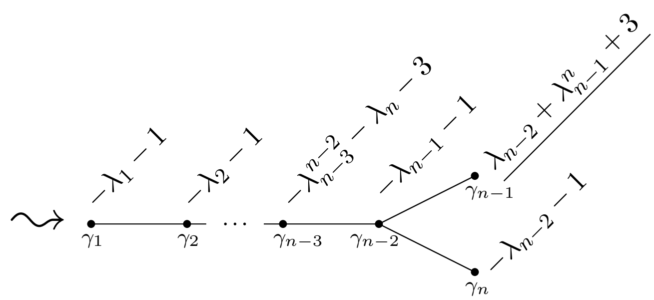
$\rightsquigarrow$ $\vdots$

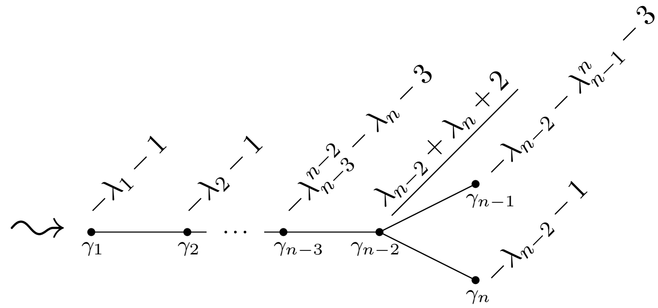
$\rightsquigarrow$

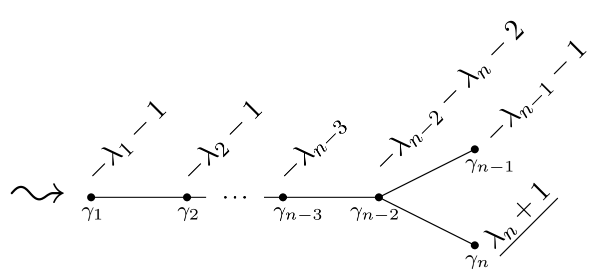


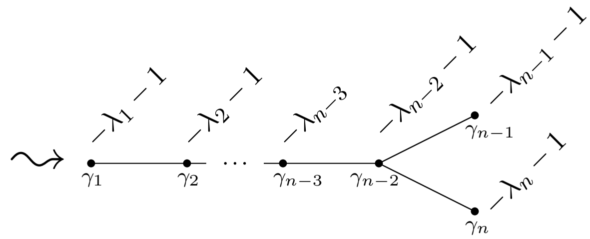

$\rightsquigarrow$
$-\lambda_1-1$
$-\lambda_2-1$
$-\lambda_{n-1}-1$
$\lambda^{n-2}_{n-3}+\lambda^{n}_{n-1}+4$
$-\lambda_{n-3}-1$
$-\lambda^{n-2}_{n-3}-\lambda^{n}_{n-2}-5$
γ_1 γ_2 $\cdots$ γ_{n-3} γ_{n-2} γ_{n-1} γ_n


$\rightsquigarrow$
$-\lambda_1-1$
$-\lambda_2-1$
$\lambda^{n-2}_{n-3}+\lambda_n+3$
$-\lambda^{n-2}_{n-3}-\lambda^{n}_{n-1}-4$
$\lambda_{n-2}+\lambda^{n}_{n-1}+3$
$-\lambda_{n-2}-1$
γ_1 γ_2 $\cdots$ γ_{n-3} γ_{n-2} γ_{n-1} γ_n


$\rightsquigarrow$
$-\lambda_1-1$
$-\lambda_2-1$
$-\lambda^{n-2}_{n-3}-\lambda_n-3$
$-\lambda_{n-1}-1$
$\lambda_{n-2}+\lambda^{n}_{n-1}+3$
$-\lambda_{n-2}-1$
γ_1 γ_2 $\cdots$ γ_{n-3} γ_{n-2} γ_{n-1} γ_n

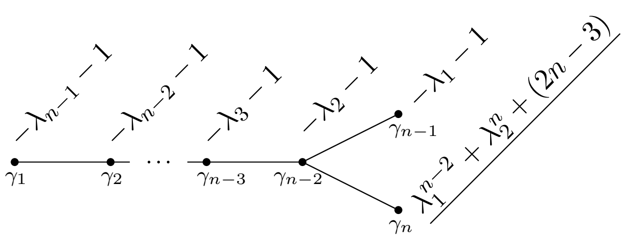

$\rightsquigarrow$
$-\lambda_1-1$
$-\lambda_2-1$
$-\lambda^{n-2}_{n-3}-\lambda_n-3$
$\lambda_{n-2}+\lambda_n+2$
$-\lambda_{n-2}-\lambda^{n}_{n-1}-3$
$-\lambda_{n-2}-1$
γ_1 γ_2 $\cdots$ γ_{n-3} γ_{n-2} γ_{n-1} γ_n

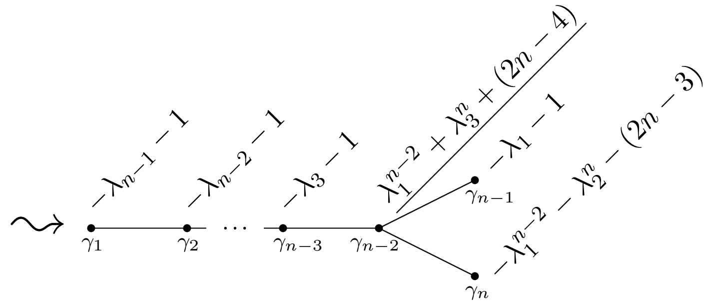

$\rightsquigarrow$
$-\lambda_1-1$
$-\lambda_2-1$
$-\lambda_{n-3}$
$-\lambda_{n-2}-\lambda_n-2$
$-\lambda_{n-1}-1$
λ_n+1
γ_1 γ_2 $\cdots$ γ_{n-3} γ_{n-2} γ_{n-1} γ_n

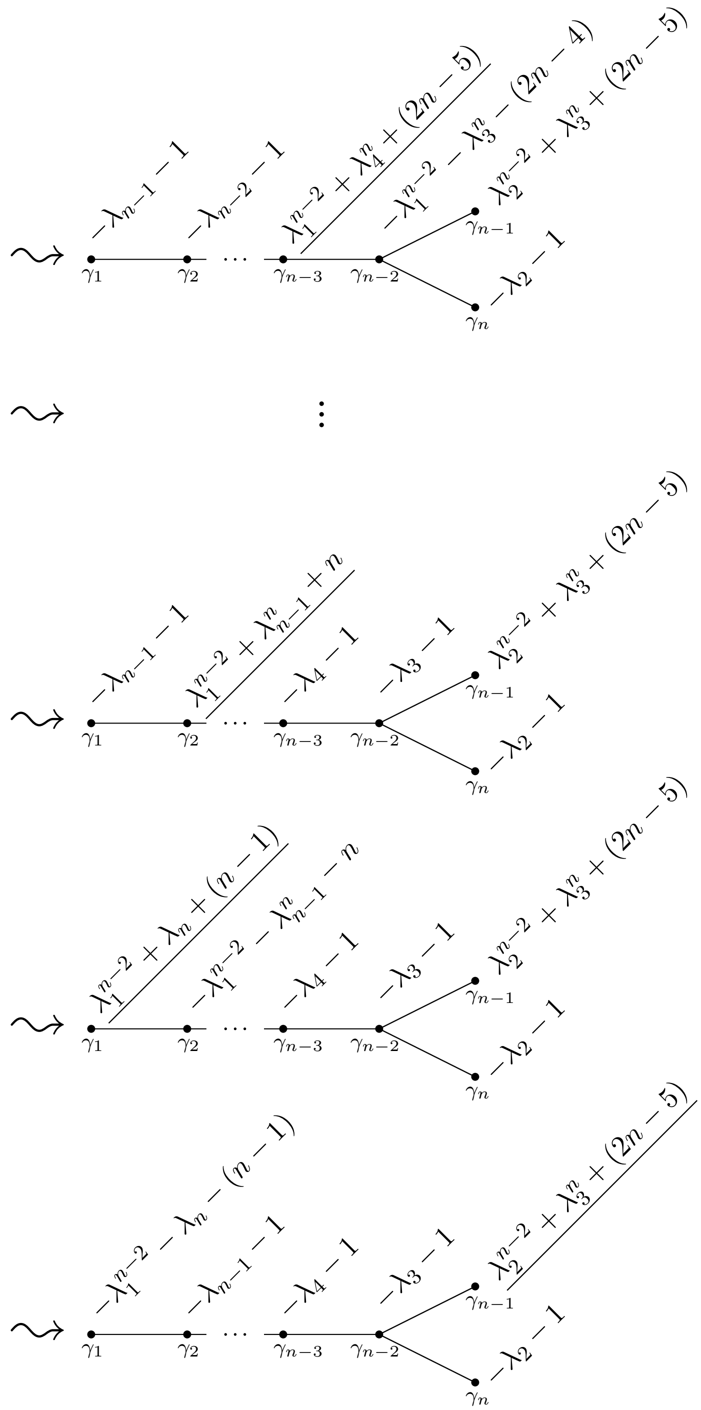


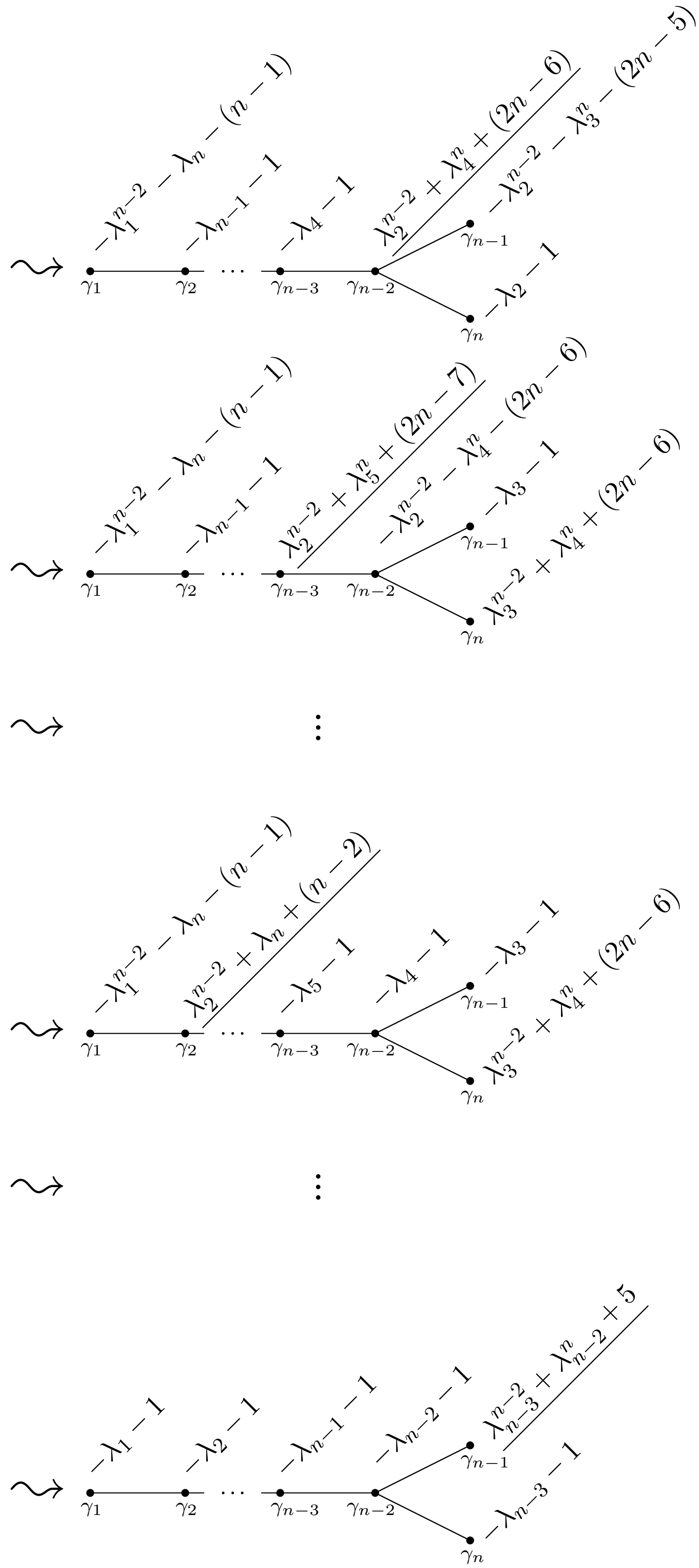
The underlined values in the above sequence do indeed match those in the second double product of the proposed $qdim_\lambda^{D_n}$, and so the Quantum Dimension Polynomial Identity holds for type D_n, where n is even.

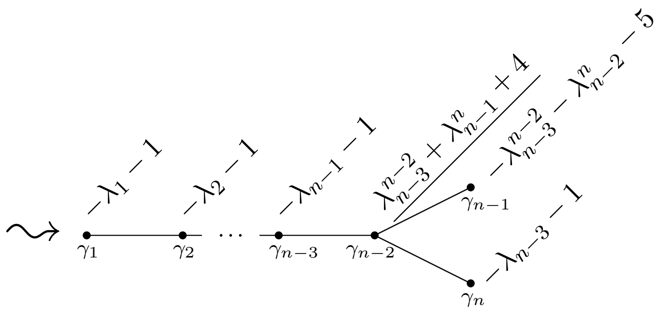
For odd n, continuing gameplay using firing sequence $(\gamma_n, \gamma_{n-2}, \gamma_{n-3}, \ldots, \gamma_2, \gamma_1; \gamma_{n-1}, \gamma_{n-2}, \gamma_{n-3}, \ldots, \gamma_2; \ldots; \gamma_{n-1}, \gamma_{n-2}, \gamma_{n-3}; \gamma_n, \gamma_{n-2}; \gamma_{n-1})$ gives us the following:

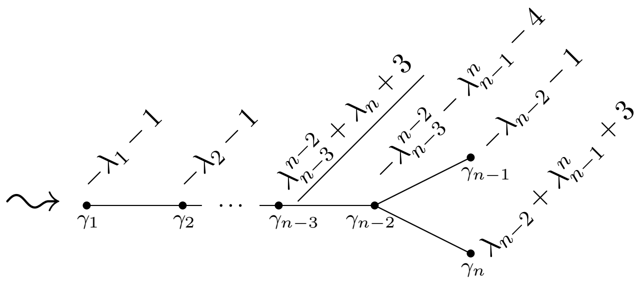


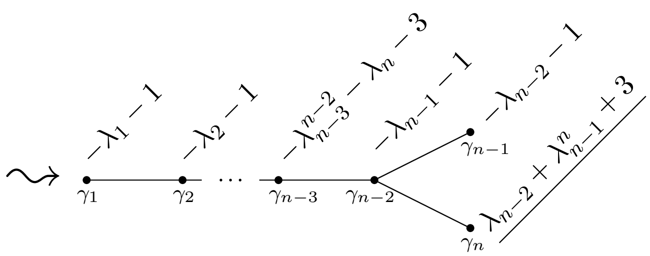


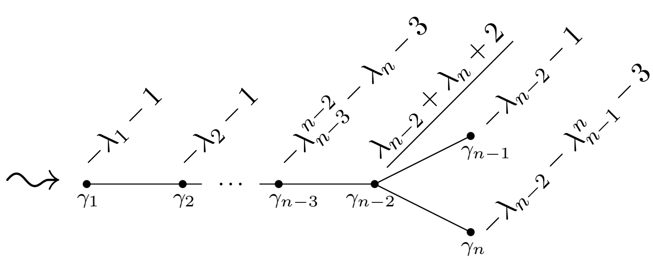

$\rightsquigarrow$
$-\lambda_{n-1}-1$
$-\lambda_{n-2}-1$
$\lambda_1^{n-2}+\lambda_4^n+(2n-5)$
$-\lambda_1^{n-2}-\lambda_3^n-(2n-4)$
$\lambda_2^{n-2}+\lambda_3^n+(2n-5)$
$-\lambda_2-1$
γ_1 γ_2 $\cdots$ γ_{n-3} γ_{n-2} γ_{n-1} γ_n
$\rightsquigarrow$ $\vdots$
$\rightsquigarrow$
$-\lambda_{n-1}-1$
$\lambda_1^{n-2}+\lambda_{n-1}^n+n$
$-\lambda_4-1$
$-\lambda_3-1$
$\lambda_2^{n-2}+\lambda_3^n+(2n-5)$
$-\lambda_2-1$
γ_1 γ_2 $\cdots$ γ_{n-3} γ_{n-2} γ_{n-1} γ_n
$\rightsquigarrow$
$\lambda_1^{n-2}+\lambda_n+(n-1)$
$-\lambda_1^{n-2}-\lambda_{n-1}^n-n$
$-\lambda_4-1$
$-\lambda_3-1$
$\lambda_2^{n-2}+\lambda_3^n+(2n-5)$
$-\lambda_2-1$
γ_1 γ_2 $\cdots$ γ_{n-3} γ_{n-2} γ_{n-1} γ_n
$\rightsquigarrow$
$-\lambda_1^{n-2}-\lambda_n-(n-1)$
$-\lambda_{n-1}-1$
$-\lambda_4-1$
$-\lambda_3-1$
$\lambda_2^{n-2}+\lambda_3^n+(2n-5)$
$-\lambda_2-1$
γ_1 γ_2 $\cdots$ γ_{n-3} γ_{n-2} γ_{n-1} γ_n

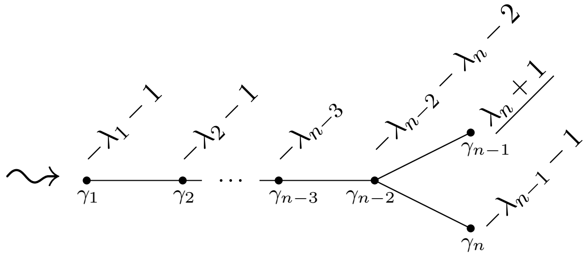

$\rightsquigarrow$
$-\lambda_1^{n-2}-\lambda_n-(n-1)$
$-\lambda_{n-1}-1$
$-\lambda_4-1$
$\lambda_2^{n-2}+\lambda_4^n+(2n-6)$
$-\lambda_2^{n-2}-\lambda_3^n-(2n-5)$
$-\lambda_2-1$
γ_1 γ_2 $\cdots$ γ_{n-3} γ_{n-2} γ_{n-1} γ_n
$\rightsquigarrow$
$-\lambda_1^{n-2}-\lambda_n-(n-1)$
$-\lambda_{n-1}-1$
$\lambda_2^{n-2}+\lambda_5^n+(2n-7)$
$-\lambda_2^{n-2}-\lambda_4^n-(2n-6)$
$-\lambda_3-1$
$\lambda_3^{n-2}+\lambda_4^n+(2n-6)$
γ_1 γ_2 $\cdots$ γ_{n-3} γ_{n-2} γ_{n-1} γ_n
$\rightsquigarrow$ $\vdots$
$\rightsquigarrow$
$-\lambda_1^{n-2}-\lambda_n-(n-1)$
$\lambda_2^{n-2}+\lambda_n+(n-2)$
$-\lambda_5-1$
$-\lambda_4-1$
$-\lambda_3-1$
$\lambda_3^{n-2}+\lambda_4^n+(2n-6)$
γ_1 γ_2 $\cdots$ γ_{n-3} γ_{n-2} γ_{n-1} γ_n
$\rightsquigarrow$ $\vdots$
$\rightsquigarrow$
$-\lambda_1-1$
$-\lambda_2-1$
$-\lambda_{n-1}-1$
$-\lambda_{n-2}-1$
$\lambda_{n-3}^{n-2}+\lambda_{n-2}^n+5$
$-\lambda_{n-3}-1$
γ_1 γ_2 $\cdots$ γ_{n-3} γ_{n-2} γ_{n-1} γ_n

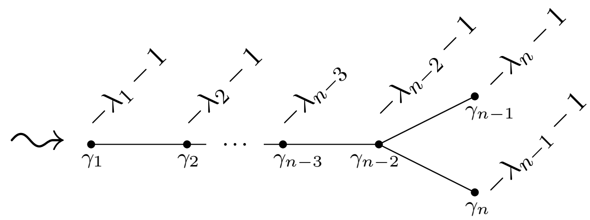

$-\lambda_1 - 1$
$-\lambda_2 - 1$
$-\lambda_{n-1} - 1$
$\overline{\lambda^{n-2}_{n-3} + \lambda^{n}_{n-1} + 4}$
$-\lambda^{n-2}_{n-3} - \lambda^{n}_{n-2} - 5$
$-\lambda_{n-3} - 1$
γ_1
γ_2
$\cdots$
γ_{n-3}
γ_{n-2}
γ_{n-1}
γ_n


$-\lambda_1 - 1$
$-\lambda_2 - 1$
$\overline{\lambda^{n-2}_{n-3} + \lambda_n + 3}$
$-\lambda^{n-2}_{n-3} - \lambda^{n}_{n-1} - 4$
$-\lambda_{n-2} - 1$
$\lambda_{n-2} + \lambda^{n}_{n-1} + 3$
γ_1
γ_2
$\cdots$
γ_{n-3}
γ_{n-2}
γ_{n-1}
γ_n


$-\lambda_1 - 1$
$-\lambda_2 - 1$
$-\lambda^{n-2}_{n-3} - \lambda_n - 3$
$-\lambda_{n-1} - 1$
$-\lambda_{n-2} - 1$
$\underline{\lambda_{n-2} + \lambda^{n}_{n-1} + 3}$
γ_1
γ_2
$\cdots$
γ_{n-3}
γ_{n-2}
γ_{n-1}
γ_n


$-\lambda_1 - 1$
$-\lambda_2 - 1$
$-\lambda^{n-2}_{n-3} - \lambda_n - 3$
$\overline{\lambda_{n-2} + \lambda_n + 2}$
$-\lambda_{n-2} - 1$
$-\lambda_{n-2} - \lambda^{n}_{n-1} - 3$
γ_1
γ_2
$\cdots$
γ_{n-3}
γ_{n-2}
γ_{n-1}
γ_n


$-\lambda_1 - 1$
$-\lambda_2 - 1$
$-\lambda_{n-3}$
$-\lambda_{n-2} - \lambda_n - 2$
$\overline{\lambda_n + 1}$
$-\lambda_{n-1} - 1$
γ_1
γ_2
$\cdots$
γ_{n-3}
γ_{n-2}
γ_{n-1}
γ_n



Again, the underlined values in the above sequence do indeed match those in the second double product of the proposed $qdim_\lambda^{D_n}$, and so the Quantum Dimension Polynomial Identity holds for type D_n, where n is odd. $\square$

3.3 Examples

To demonstrate what Quantum Dimension Polynomials tend to look like, we present the following examples of types A_n and D_n, respectively.

Example 3.3.1. Let $\mathscr{G} = A_3$ and let $\lambda = (2, 1, 0)$. Then

$$\begin{aligned}
qdim_\lambda^{A_3} &= \frac{[\lambda_1+1]_q}{[1]_q}\frac{[\lambda_1+\lambda_2+2]_q}{[2]_q}\frac{[\lambda_1+\lambda_2+\lambda_3+3]_q}{[3]_q}\frac{[\lambda_2+1]_q}{[1]_q}\frac{[\lambda_2+\lambda_3+2]_q}{[2]_q}\frac{[\lambda_3+1]_q}{[1]_q}\\
&= \frac{[3]_q}{[1]_q}\frac{[5]_q}{[2]_q}\frac{[6]_q}{[3]_q}\frac{[2]_q}{[1]_q}\frac{[3]_q}{[2]_q}\frac{[1]_q}{[1]_q}\\
&= \frac{(1-q^3)(1-q^5)(1-q^6)(1-q^2)(1-q^3)(1-q)}{(1-q)(1-q^2)(1-q^3)(1-q)(1-q^2)(1-q)}\\
&= \frac{1-q^3}{1-q}\cdot\frac{1-q^5}{1-q}\cdot\frac{1-q^6}{1-q^2}
\end{aligned}$$

$$= (1+q+q^2)(1+q+q^2+q^3+q^4)(1+q^2+q^4)$$
$$= 1+2q+4q^2+5q^3+7q^4+7q^5+7q^6+5q^7+4q^8+2q^9+q^{10}.$$

Example 3.3.2. Let $\mathscr{G} = D_4$ and let $\lambda = (2,1,0,1)$. Note that from Example 3.3.1,

$$\prod_{i=1}^{3}\prod_{j=i}^{3} \frac{[\lambda_i^j + j + 1 - i]_q}{[j+1-i]_q} = qdim^{A_3}_{(2,1,0)}.$$

Thus,

$$\begin{aligned}
qdim^{D_4}_{\lambda} &= qdim^{A_3}_{(2,1,0)} \cdot \frac{[\lambda_1^2+\lambda_2^4+5]_q}{[5]_q} \frac{[\lambda_1^2+\lambda_3^4+4]_q}{[4]_q} \frac{[\lambda_1^2+\lambda_4+3]_q}{[3]_q} \\
&\quad \cdot \frac{[\lambda_2+\lambda_3^4+3]_q}{[3]_q} \frac{[\lambda_2+\lambda_4+2]_q}{[2]_q} \frac{[\lambda_4+1]_q}{[1]_q} \\
&= qdim^{A_3}_{(2,1,0)} \cdot \frac{[10]_q}{[5]_q} \frac{[8]_q}{[4]_q} \frac{[7]_q}{[3]_q} \frac{[5]_q}{[3]_q} \frac{[4]_q}{[2]_q} \frac{[2]_q}{[1]_q} \\
&= qdim^{A_3}_{(2,1,0)} \cdot \frac{(1-q^{10})(1-q^8)(1-q^7)(1-q^5)(1-q^4)(1-q^2)}{(1-q^5)(1-q^4)(1-q^3)(1-q^3)(1-q^2)(1-q)} \\
&= qdim^{A_3}_{(2,1,0)} \cdot \frac{(1-q^{10})(1-q^8)(1-q^7)}{(1-q^3)(1-q^3)(1-q)} \\
&= \frac{(1-q^3)(1-q^5)(1-q^6)}{(1-q)(1-q)(1-q^2)} \cdot \frac{(1-q^{10})(1-q^8)(1-q^7)}{(1-q^3)(1-q^3)(1-q)} \\
&= \frac{(1-q^5)}{(1-q)} \cdot \frac{(1-q^6)}{(1-q^3)} \cdot \frac{(1-q^{10})}{(1-q^2)} \cdot \frac{(1-q^8)}{(1-q)} \cdot \frac{(1-q^7)}{(1-q)} \\
&= (1+q+q^2+q^3+q^4)(1+q^3)(1+q^2+q^4+q^6+q^8) \\
&\quad \cdot (1+q+q^2+q^3+q^4+q^5+q^6+q^7)(1+q+q^2+q^3+q^4+q^5+q^6).
\end{aligned}$$

(Note that this is a polynomial of degree 28.)

Chapter 4

Applications

Now that we have stated and proved the Quantum Dimension Polynomial Identity and examined a few examples, we now provide a few applications. First, in Section 4.1 we provide an identity for $qdim^{A_n}_{\omega_k}$. Second, in Section 4.2 we use QDPs to answer a combinatorial problem involving the number of possible subsets of a given set that satisfy a certain property.

4.1 A special identity for $qdim^{A_n}_{\omega_k}$

Recall that for a positive integer $k \leq n$, the n-tuple ω_k is the k-th fundamental weight with the value 1 in the k-th position and the value 0 elsewhere. For example, for $n = 6$, $\omega_4 = (0, 0, 0, 1, 0, 0)$.

What happens if we examine $qdim^{A_n}_{w_k}$? It turns out that this QDP can be expressed in a very nice way. But first, let us declare some notation.

Definition. For an integer $n \geq 1$, we extend the factorial operator ! to q-integers by defining

$$[n]_q! := [n]_q[n-1]_q \cdots [1]_q.$$

Additionally, for an integer k such that $1 \leq k \leq n$, we extend the typical *falling factorial* notation, which takes the first k factors in the factorial as follows:

$$([n]_q)_k = [n]_q[n-1]_q \cdots [(n-k)+1]_q = \frac{[n]_q!}{[n-k]_q!}.$$

Lastly, when $0 \leq k \leq n$ we extend the 'n choose k' operator to q-integers by defining

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{([n]_q)_k}{[k]_q!}.$$

Example 4.1.1.

$$\begin{aligned}
\binom{5}{3}_q &= \frac{([5]_q)_3}{[3]_q!} \\
&= \frac{(1-q^5)(1-q^4)(1-q^3)}{(1-q^3)(1-q^2)(1-q)} \\
&= \frac{1-q^5}{1-q} \cdot \frac{1-q^4}{1-q^2} \\
&= (1+q+q^2+q^3+q^4)(1+q^2) \\
&= 1+q+2q^2+2q^3+2q^4+q^5+q^6.
\end{aligned}$$

Note that when evaluated at $q = 1$,

$$\left.\binom{5}{3}_q\right|_{q=1} = 10 = \binom{5}{3}.$$

We are now ready to state the following proposition:

Proposition 4.1.1. *For positive integers k and n with $k \leq n$,*

$$qdim_{\omega_k}^{A_n} = \binom{n+1}{k}_q,$$

a degree $k(n+1-k)$ polynomial.

Proof. Let k and n be positive integers with $k \leq n$. From Theorem 3,

$$qdim_{\omega_k}^{A_n} = \prod_{i=1}^{n} \prod_{j=i}^{n} \frac{[\lambda_i^j + j + 1 - i]_q}{[j+1-i]_q}.$$

Note that $\lambda_i^j = 1$ if $i \leq k \leq j$ and $\lambda_i^j = 0$ otherwise. Thus, we get the following series of equalities involving a long double product of q-integer fractions that each telescope:

$$
\begin{aligned}
qdim_{\omega_k}^{A_n} &= \prod_{i=1}^{n}\prod_{j=i}^{n}\frac{[\lambda_i^j + j + 1 - i]_q}{[j+1-i]_q} \\
&= \prod_{i=1}^{k}\prod_{j=k}^{n}\frac{[j+2-i]_q}{[j+1-i]_q} \\
&= \left(\frac{[k+2-1]_q}{[k+1-1]_q}\cdot\frac{[k+3-1]_q}{[k+2-1]_q}\cdot\frac{[k+4-1]_q}{[k+3-1]_q}\cdot\ \cdots\ \cdot\frac{[n+2-1]_q}{[n+1-1]_q}\right) \\
&\quad\cdot\left(\frac{[k+2-2]_q}{[k+1-2]_q}\cdot\frac{[k+3-2]_q}{[k+2-2]_q}\cdot\frac{[k+4-2]_q}{[k+3-2]_q}\cdot\ \cdots\ \cdot\frac{[n+2-2]_q}{[n+1-2]_q}\right) \\
&\quad\cdots \qquad (\star) \\
&\quad\cdot\left(\frac{[k+2-k]_q}{[k+1-k]_q}\cdot\frac{[k+3-k]_q}{[k+2-k]_q}\cdot\frac{[k+4-k]_q}{[k+3-k]_q}\cdot\ \cdots\ \cdot\frac{[n+2-k]_q}{[n+1-k]_q}\right) \\
&= \left(\frac{[n+2-1]_q}{[k+1-1]_q}\right)\cdot\left(\frac{[n+2-2]_q}{[k+1-2]_q}\right)\cdot\ \cdots\ \cdot\left(\frac{[n+2-k]_q}{[k+1-k]_q}\right) \\
&= \frac{[n+1]_q[n]_q\cdots[(n+1)+1-k]_q}{[k]_q[k-1]_q\cdots[1]_q} \\
&= \frac{([n+1]_q)_k}{[k]_q!} \\
&= \binom{n+1}{k}_q.
\end{aligned}
$$

Interestingly, note that in the telescoping double product at $(\star)$ there are $k(n+1-k)$ total fractions. Moreover, the degree of the resulting polynomial is $k(n+1-k)$, as there are, after cancellation, k fractions multiplied together, each contributing a degree $n+1-k$ polynomial. $\square$

Looking back at Example 4.1.1, note that the degree of $\binom{5}{3}_q$ is $6 = 3(4+1-3) = k(n+1-k)$.

Example 4.1.2. To see the telescoping products in action, we will examine $qdim^{A_5}_{\omega_k}$ for all k. We break up each single product by parentheses, noting that in each block every q-integer cancels out except for the last numerator and the first denominator.

$$\begin{aligned}
qdim^{A_5}_{\omega_1} &= \frac{[\lambda_1^1+1]_q}{[1]_q}\cdot\frac{[\lambda_1^2+2]_q}{[2]_q}\cdot\frac{[\lambda_1^3+3]_q}{[3]_q}\cdot\frac{[\lambda_1^4+4]_q}{[4]_q}\cdot\frac{[\lambda_1^5+5]_q}{[5]_q}\\
&= \frac{[2]_q}{[1]_q}\cdot\frac{[3]}{[2]_q}\cdot\frac{[4]_q}{[3]_q}\cdot\frac{[5]_q}{[4]_q}\cdot\frac{[6]_q}{[5]_q}\\
&= \frac{[6]_q}{[1]_q} = \frac{([6]_q)_1}{[1]_q} = \binom{6}{1}_q.
\end{aligned}$$

$$\begin{aligned}
qdim^{A_5}_{\omega_2} &= \left(\frac{[\lambda_1^2+2]_q}{[2]_q}\cdot\frac{[\lambda_1^3+3]_q}{[3]_q}\cdot\frac{[\lambda_1^4+4]_q}{[4]_q}\cdot\frac{[\lambda_1^5+5]_q}{[5]_q}\right)\\
&\quad\cdot\left(\frac{[\lambda_2^2+1]_q}{[1]_q}\cdot\frac{[\lambda_2^3+2]_q}{[2]_q}\cdot\frac{[\lambda_2^4+3]_q}{[3]_q}\cdot\frac{[\lambda_2^5+4]_q}{[4]_q}\right)\\
&= \left(\frac{[3]_q}{[2]_q}\cdot\frac{[4]_q}{[3]_q}\cdot\frac{[5]_q}{[4]_q}\cdot\frac{[6]_q}{[5]_q}\right)\\
&\quad\cdot\left(\frac{[2]_q}{[1]_q}\cdot\frac{[3]_q}{[2]_q}\cdot\frac{[4]_q}{[3]_q}\cdot\frac{[5]_q}{[4]_q}\right)\\
&= \frac{[6]_q[5]_q}{[2]_q[1]_q} = \frac{([6]_q)_2}{[2]_q} = \binom{6}{2}_q.
\end{aligned}$$

$$
\begin{aligned}
qdim^{A_5}_{\omega_3} &= \left(\frac{[\lambda_1^3+3]_q}{[3]_q}\cdot\frac{[\lambda_1^4+4]_q}{[4]_q}\cdot\frac{[\lambda_1^5+5]_q}{[5]_q}\right)\\
&\quad\cdot\left(\frac{[\lambda_2^3+2]_q}{[2]_q}\cdot\frac{[\lambda_2^4+3]_q}{[3]_q}\cdot\frac{[\lambda_2^5+4]_q}{[4]_q}\right)\\
&\quad\cdot\left(\frac{[\lambda_3^3+1]_q}{[1]_q}\cdot\frac{[\lambda_3^4+2]_q}{[2]_q}\cdot\frac{[\lambda_3^5+3]_q}{[3]_q}\right)\\
&= \left(\frac{[4]_q}{[3]_q}\cdot\frac{[5]_q}{[4]_q}\cdot\frac{[6]_q}{[5]_q}\right)\\
&\quad\cdot\left(\frac{[3]_q}{[2]_q}\cdot\frac{[4]_q}{[3]_q}\cdot\frac{[5]_q}{[4]_q}\right)\\
&\quad\cdot\left(\frac{[2]_q}{[1]_q}\cdot\frac{[3]_q}{[2]_q}\cdot\frac{[4]_q}{[3]_q}\right)\\
&= \frac{[6]_q[5]_q[4]_q}{[3]_q[2]_q[1]_q} = \frac{([6]_q)_3}{[3]_q} = \binom{6}{3}_q.
\end{aligned}
$$

$$
\begin{aligned}
qdim^{A_5}_{\omega_4} &= \left(\frac{[\lambda_1^4+4]_q}{[4]_q}\cdot\frac{[\lambda_1^5+5]_q}{[5]_q}\right)\cdot\left(\frac{[\lambda_2^4+3]_q}{[3]_q}\cdot\frac{[\lambda_2^5+4]_q}{[4]_q}\right)\\
&\quad\cdot\left(\frac{[\lambda_3^4+2]_q}{[2]_q}\cdot\frac{[\lambda_3^5+3]_q}{[3]_q}\right)\cdot\left(\frac{[\lambda_4^4+1]_q}{[1]_q}\cdot\frac{[\lambda_4^5+2]_q}{[2]_q}\right)\\
&= \left(\frac{[5]_q}{[4]_q}\cdot\frac{[6]_q}{[5]_q}\right)\cdot\left(\frac{[4]_q}{[3]_q}\cdot\frac{[5]_q}{[4]_q}\right)\\
&\quad\cdot\left(\frac{[3]_q}{[2]_q}\cdot\frac{[4]_q}{[3]_q}\right)\cdot\left(\frac{[2]_q}{[3]_q}\cdot\frac{[3]_q}{[2]_q}\right)\\
&= \frac{[6]_q[5]_q[4]_q[3]_q}{[4]_q[3]_q[2]_q[1]_q} = \frac{([6]_q)_4}{[4]_q} = \binom{6}{4}_q.
\end{aligned}
$$

$$
\begin{aligned}
qdim^{A_5}_{\omega_5} &= \left(\frac{[\lambda_1^5+5]_q}{[5]_q}\right)\cdot\left(\frac{[\lambda_2^5+4]_q}{[4]_q}\right)\cdot\left(\frac{[\lambda_3^5+3]_q}{[3]_q}\right)\cdot\left(\frac{[\lambda_4^5+2]_q}{[2]_q}\right)\cdot\left(\frac{[\lambda_5^5+1]_q}{[1]_q}\right)\\
&= \left(\frac{[6]_q}{[5]_q}\right)\cdot\left(\frac{[5]_q}{[4]_q}\right)\cdot\left(\frac{[4]_q}{[3]_q}\right)\cdot\left(\frac{[3]_q}{[2]_q}\right)\cdot\left(\frac{[2]_q}{[1]_q}\right)\\
&= \frac{([6]_q)_5}{[5]_q} = \binom{6}{5}_q.
\end{aligned}
$$

4.2 The lattice of k-element subsets of an $(n+1)$-element set

Consider the following combinatorial problem: given the set $A = \{1, 2, \ldots, n+1\}$, how many k-element subsets of A are there whose elements sum to a fixed number s? A version of this problem was posed in [Proc1] and addressed using ideas similar to what we present next. In the following, we largely assume knowledge of distributive lattices.

To approach this problem using QDPs, we will construct a lattice of k-element subsets of A. To this end, consider a k-element subset $S \subseteq A$ as a strictly increasing k-tuple $\{s_1, s_2, \ldots, s_k\}$. Given another such subset $T = \{t_1, t_2, \ldots, t_k\}$, say $S \leq T \iff s_i \leq t_i$ for each i. Let $L_n(k)$ be the set of all k-element subsets with respect to this partial order.

Note that for any $S, T \in L_n(k)$, there is a least upper bound $S \vee T = \{max(s_1, t_1), \ldots, max(s_k, t_k)\}$ and a greatest lower bound $S \wedge T = \{min(s_1, t_1), \ldots, min(s_k, t_k)\}$. Additionally, '$\vee$' distributes over '$\wedge$' and vice-versa:

$$R \vee (S \wedge T) = (R \vee S) \wedge (R \vee T)$$

and

$$R \wedge (S \vee T) = (R \wedge S) \vee (R \wedge T).$$

That is, $L_n(k)$ is a distributive lattice.

With respect to this lattice ordering, the minimal element (with rank

0) is $\{1, \ldots, k\}$ and the maximal element (with rank $k \cdot (n+1-k)$) is $\{n+2-k, \ldots, n, n+1\}$. Also, the rank of any $S \in L_n(k)$ is

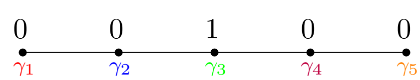
$$\rho(S) = s_1 + \cdots + s_k - (1 + \cdots + k) = s_1 + \cdots + s_k - \frac{k(k+1)}{2}.$$

Now, for $T \in L_n(k)$, T covers S, written $S \longrightarrow T$, if and only if there is some $j \in \{1, \ldots, k\}$ such that $s_j + 1 = t_j$, while $s_i = t_i$ whenever $i \neq j$. In this case, let $c := s_j$ ('c' is for color) and write $S \xrightarrow{c} T$.

It is a well-known fact from the theory of crystal graphs that $L_n(k)$ is the crystal graph associated with the dominant weight ω_k for the Coxeter-Dynkin flower A_n. Within this crystalline context, $L_n(k)$ is known as a 'miniscule lattice.' By the Quantum Dimension Polynomial Identity and Proposition 4.1.1,

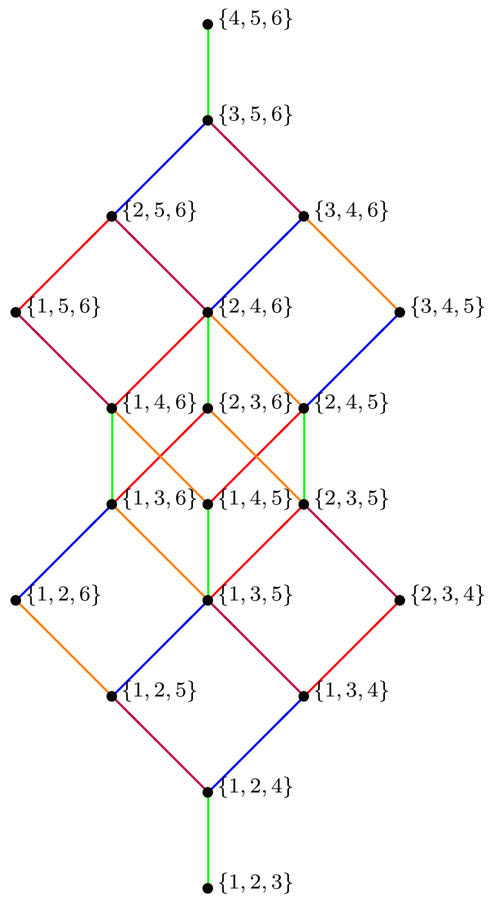
$$\binom{n+1}{k}_q = \frac{([n]_q)_k}{[k]_q!} = qdim_{\omega_k}^{A_n} = RGF(L_n(k), q).$$

Example 4.2.1. Consider the lattice $L_5(3)$, constructed as described above. This lattice can be described using A_5 with initial weight $\omega_3 = (0, 0, 1, 0, 0)$:



The lattice $L_5(3)$ is shown below:

$\underline{RGF(L_5(3), q)}$:



$$q^9 + q^8 + 2q^7 + 3q^6 + 3q^5 + 3q^4 + 3q^3 + 2q^2 + q + 1$$

Indeed, the Rank Generating Function shown on the right of $L_5(3)$ matches $qdim^{A_5}_{\omega_3}$ from Example 4.1.2 when multiplied out.

Interestingly, there is no known product formula for the number of k-element subsets of $\{1, 2, \ldots, n+1\}$ whose sum is a fixed number s. This number can be discerned as the coefficient for $q^r = q^{s - \frac{k(k+1)}{2}}$ in the q-binomial coefficient $\binom{n+1}{k}_q$ [Proc1].

For example, in the above lattice, the 3-element subsets of $\{1, 2, 3, 4, 5, 6\}$

that sum to $s = 11$ are $\{1, 4, 6\}$, $\{2, 3, 6\}$, and $\{2, 4, 5\}$, which can be found in the row with rank $s - \frac{k(k+1)}{2} = 11 - \frac{3(4)}{2} = 5$.

Chapter 5

Concluding remarks

We were able to introduce the Networked-Numbers Game for a game graph $\mathscr{G} = (\Gamma, A)$ and state that $\mathscr{G}$ is game-gratifying if and only if it is a Coxeter-Dynkin flower, as defined in Theorem 1 (La Florado Klasado). Next, we introduced Quantum Dimension Polynomials, proved a novel result involving closed-form expressions of QDPs of types A_n, B_n, C_n, and D_n, and provided a few examples of QDPs. Lastly, we proved a nice identity for $qdim_{\omega_k}^{A_n}$ and applied some QDPs of type A_n to a combinatorial problem involving enumeration of certain k-element subsets of the set $\{1, 2, \ldots, n+1\}$. These statements and results demonstrate the power and beauty of some interesting algebraic and combinatorial structures.

There are a few immediate possible directions to continue. First, do nice closed-form expressions for the QDPS of types E_6, E_7, E_8, F_4, or G_2–namely, the remaining integral Coxeter-Dynkin flowers of Figure 2.1–exist? Second, what other applications of QDPS are there? [Fis] and [Proc2] expose and enumerate certain kinds of "symmetric plane partitions"–arrays of integers that can be thought of as cubes stacked in towers above the arrays' positions–using QDPs of types B_n and C_n. Are there other kinds of symmetric plane partitions described by the QDPs of type D_n?

Appendix A

Proofs for QDPs of types B_n and C_n

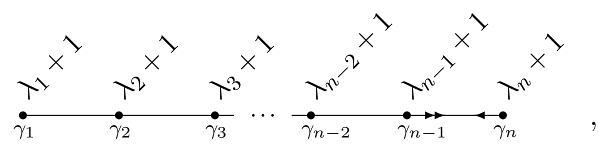
Here we provide proofs of Cases B_n and C_n of the Quantum Dimension Polynomial Identity (Theorem 3 of Section 2.2).

Proof. **(Case B_n):**

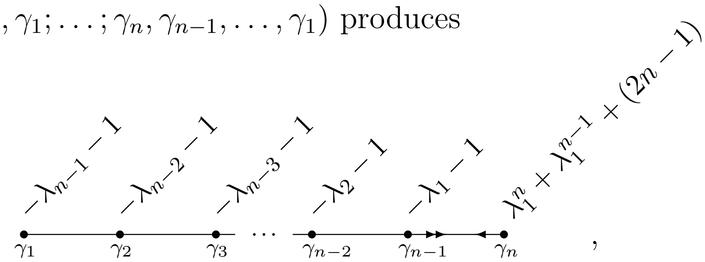
Consider our goal:

$$qdim_\lambda^{B_n} = \prod_{i=1}^{n-1}\prod_{j=i}^{n-1}\frac{[\lambda_i^j + j + 1 - i]_q}{[j+1-i]_q}\prod_{i=1}^{n}\prod_{j=i}^{n}\frac{[\lambda_i^n + \lambda_j^{n-1} + 2n + 1 - i - j]_q}{[2n+1-i-j]_q}. \tag{A.1}$$

Note that $\displaystyle\prod_{i=1}^{n-1}\prod_{j=i}^{n-1}\frac{[\lambda_i^j + j + 1 - i]_q}{[j+1-i]_q} = qdim_\lambda^{A_{n-1}}$. This falls in line with the fact that when viewing B_n as follows,

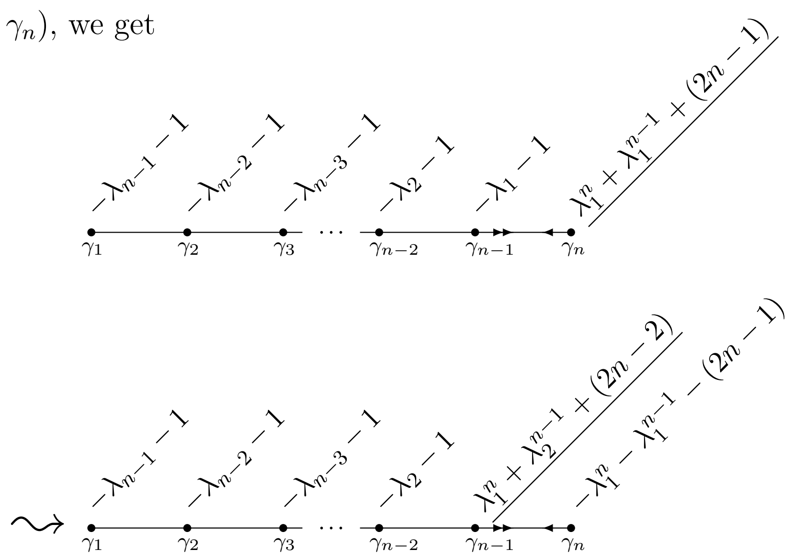


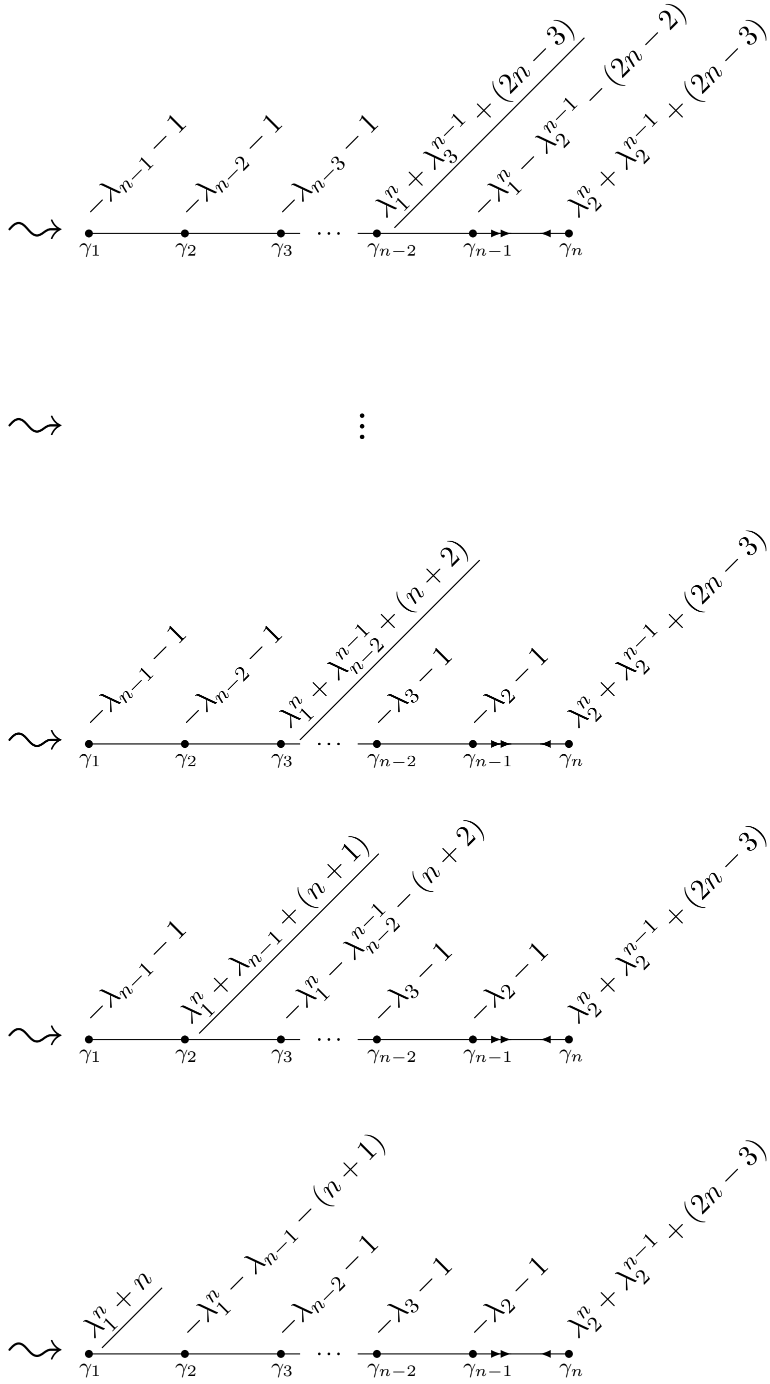
removing γ_n leaves a copy of A_{n-1}. In fact, starting with firing sequence $(\gamma_1; \gamma_2, \gamma_1; \gamma_3, \gamma_2, \gamma_1; \ldots; \gamma_n, \gamma_{n-1}, \ldots, \gamma_1)$ produces

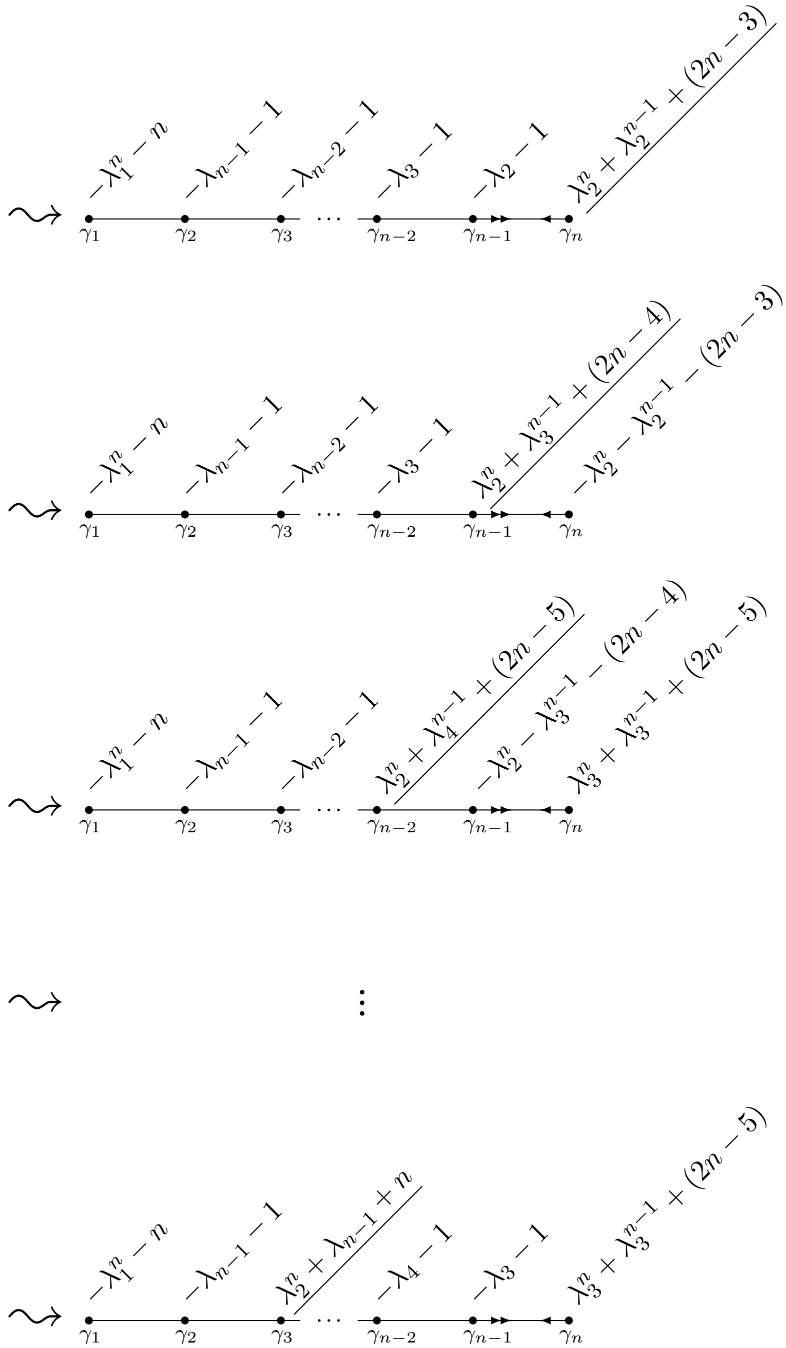


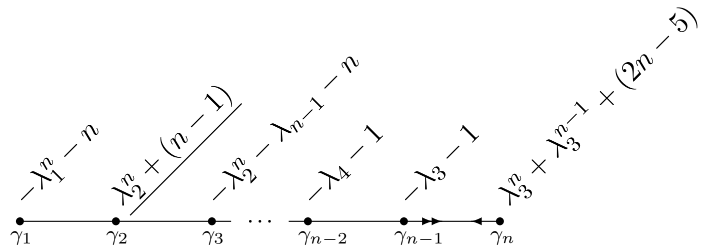
since the only time γ_{n-1} is fired it has population $\lambda_1^{n-1} + (n-1)$ and is amplified by a factor of 2 before being added to $(\lambda_n + 1)$ at γ_n.

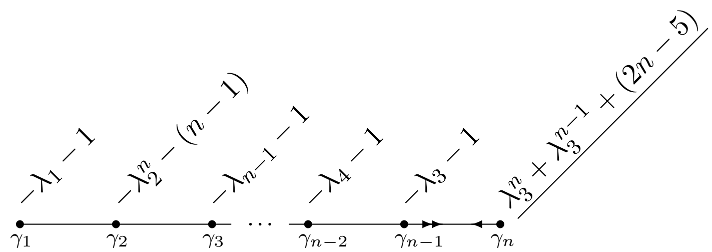
All that is left at this point is the second double product of equation A.1. Continuing gameplay using firing sequence $(\gamma_n, \gamma_{n-1}, \ldots, \gamma_1; \gamma_n, \gamma_{n-1}, \ldots, \gamma_2; \ldots; \gamma_n, \gamma_{n-1}; \gamma_n)$, we get

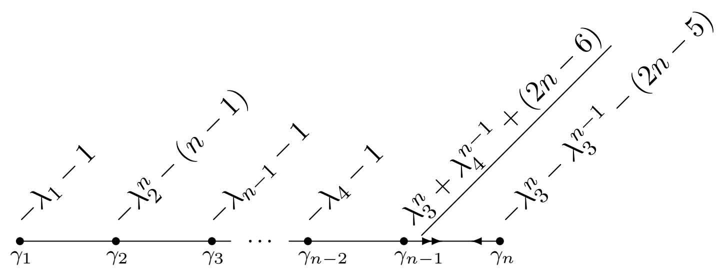


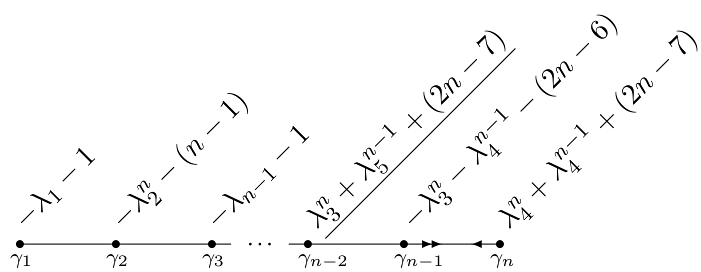

$-\lambda_{n-1}-1$
$-\lambda_{n-2}-1$
$-\lambda_{n-3}-1$
$\lambda_1^n+\lambda_3^{n-1}+(2n-3)$
$-\lambda_1^n-\lambda_2^{n-1}-(2n-2)$
$\lambda_2^n+\lambda_2^{n-1}+(2n-3)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n
$\vdots$
$-\lambda_{n-1}-1$
$-\lambda_{n-2}-1$
$\lambda_1^n+\lambda_{n-2}^{n-1}+(n+2)$
$-\lambda_3-1$
$-\lambda_2-1$
$\lambda_2^n+\lambda_2^{n-1}+(2n-3)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n
$-\lambda_{n-1}-1$
$\lambda_1^n+\lambda_{n-1}+(n+1)$
$-\lambda_1^n-\lambda_{n-2}^{n-1}-(n+2)$
$-\lambda_3-1$
$-\lambda_2-1$
$\lambda_2^n+\lambda_2^{n-1}+(2n-3)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n
λ_1^n+n
$-\lambda_1^n-\lambda_{n-1}-(n+1)$
$-\lambda_{n-2}-1$
$-\lambda_3-1$
$-\lambda_2-1$
$\lambda_2^n+\lambda_2^{n-1}+(2n-3)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n


$-\lambda_1^n - n$
$-\lambda_{n-1} - 1$
$-\lambda_{n-2} - 1$
$-\lambda_3 - 1$
$-\lambda_2 - 1$
$\lambda_2^n + \lambda_2^{n-1} + (2n-3)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n
$-\lambda_1^n - n$
$-\lambda_{n-1} - 1$
$-\lambda_{n-2} - 1$
$-\lambda_3 - 1$
$\lambda_2^n + \lambda_3^{n-1} + (2n-4)$
$-\lambda_2^n - \lambda_2^{n-1} - (2n-3)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n
$-\lambda_1^n - n$
$-\lambda_{n-1} - 1$
$-\lambda_{n-2} - 1$
$\lambda_2^n + \lambda_4^{n-1} + (2n-5)$
$-\lambda_2^n - \lambda_3^{n-1} - (2n-4)$
$\lambda_3^n + \lambda_3^{n-1} + (2n-5)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n
$\vdots$
$-\lambda_1^n - n$
$-\lambda_{n-1} - 1$
$\lambda_2^n + \lambda_{n-1} + n$
$-\lambda_4 - 1$
$-\lambda_3 - 1$
$\lambda_3^n + \lambda_3^{n-1} + (2n-5)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n

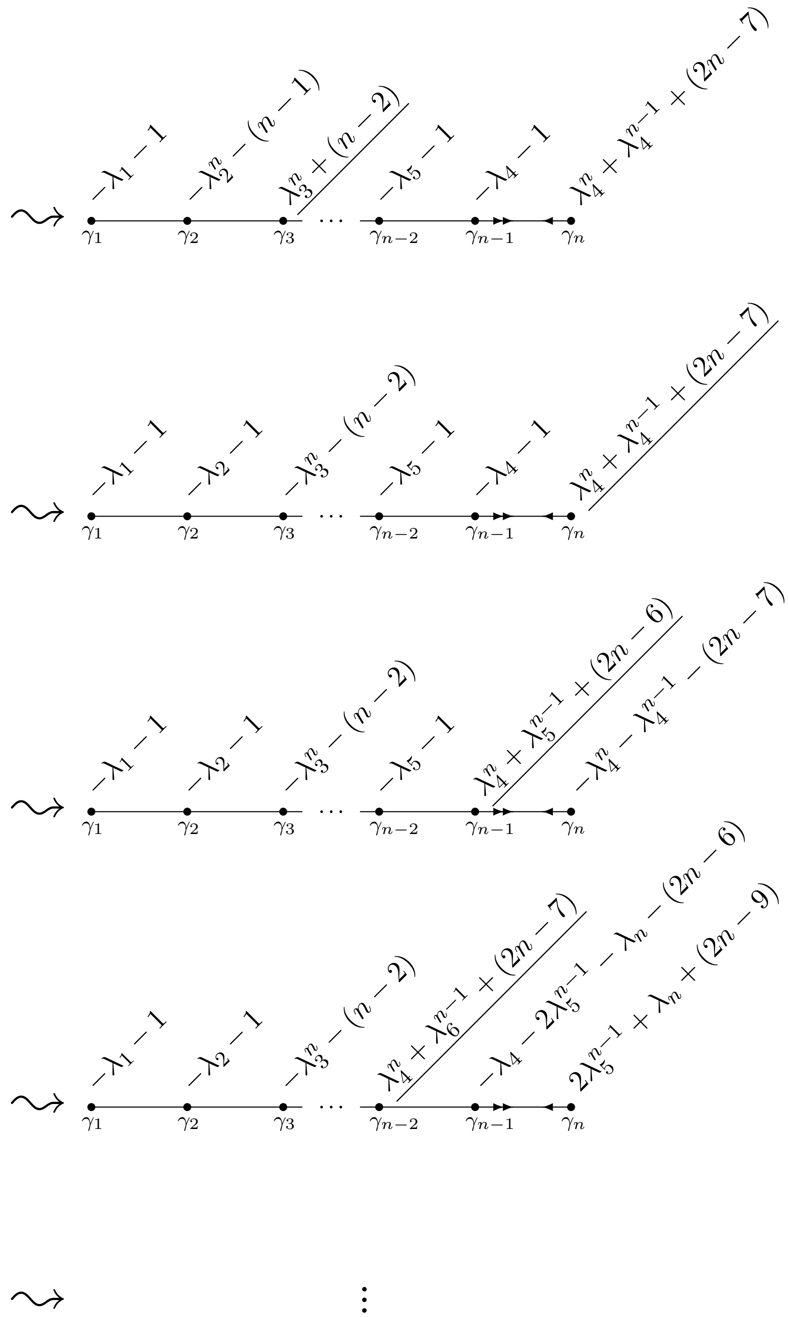
$\rightsquigarrow$ 

$\rightsquigarrow$ 

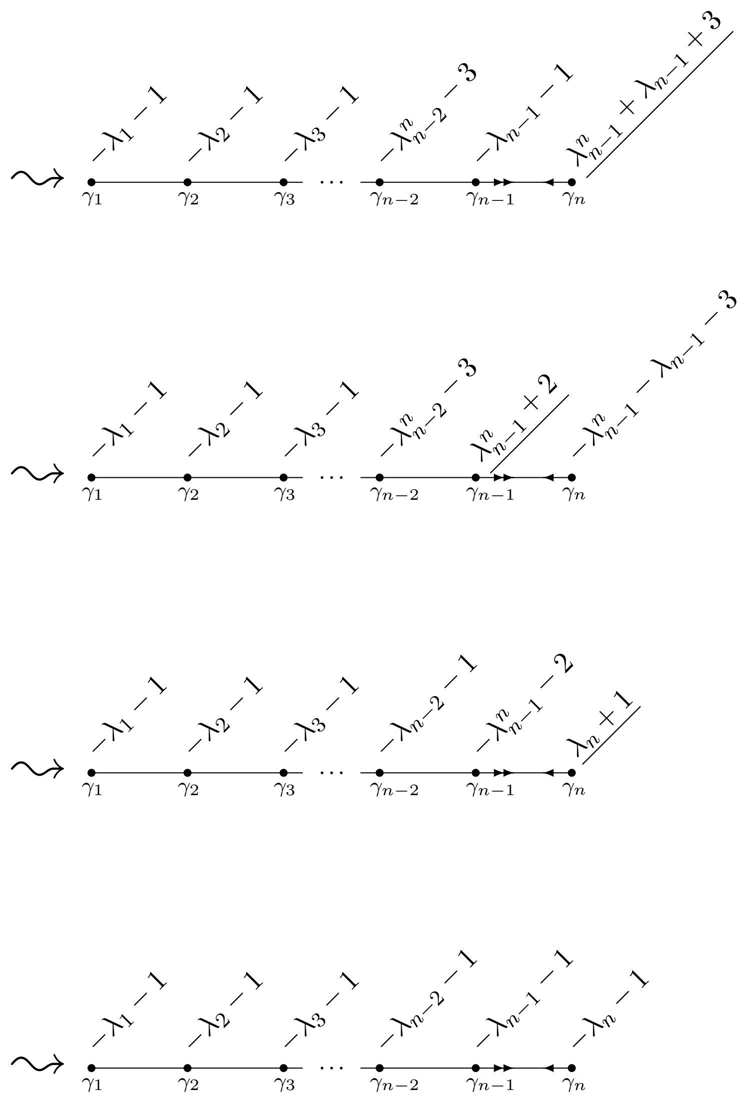
$\rightsquigarrow$ 

$\rightsquigarrow$ 

$\rightsquigarrow$ $\vdots$



$-\lambda_1-1$
$-\lambda_2^n-(n-1)$
$\lambda_3^n+(n-2)$
$-\lambda_5-1$
$-\lambda_4-1$
$\lambda_4^n+\lambda_4^{n-1}+(2n-7)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n
$-\lambda_1-1$
$-\lambda_2-1$
$-\lambda_3^n-(n-2)$
$-\lambda_5-1$
$-\lambda_4-1$
$\lambda_4^n+\lambda_4^{n-1}+(2n-7)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n
$-\lambda_1-1$
$-\lambda_2-1$
$-\lambda_3^n-(n-2)$
$-\lambda_5-1$
$\lambda_4^n+\lambda_5^{n-1}+(2n-6)$
$-\lambda_4^n-\lambda_4^{n-1}-(2n-7)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n
$-\lambda_1-1$
$-\lambda_2-1$
$-\lambda_3^n-(n-2)$
$\lambda_4^n+\lambda_6^{n-1}+(2n-7)$
$-\lambda_4-2\lambda_5^{n-1}-\lambda_n-(2n-6)$
$2\lambda_5^{n-1}+\lambda_n+(2n-9)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n



The underlined values in the above sequence do indeed match those in the second double product of equation A.1, and so the Quantum Dimension Polynomial Identity holds for type B_n.

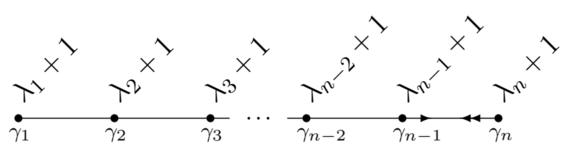
(Case C_n):

Consider our goal:

$$qdim_{\lambda}^{C_n} = \prod_{i=1}^{n-1}\prod_{j=i}^{n-1} \frac{[\lambda_i^j + j + 1 - i]_q}{[j+1-i]_q} \prod_{i=1}^{n}\prod_{j=i+1}^{n+1} \frac{[\lambda_i^n + \lambda_j^n + 2n + 2 - i - j]_q}{[2n+2-i-j]_q}. \tag{A.2}$$

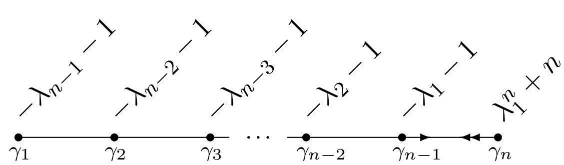
The proof proceeds similarly to the above of type B_n.

Note that $\displaystyle\prod_{i=1}^{n-1}\prod_{j=i}^{n-1} \frac{[\lambda_i^j + j + 1 - i]_q}{[j+1-i]_q} = qdim_{\lambda}^{A_{n-1}}$. This falls in line with the fact that when viewing C_n as follows,



,

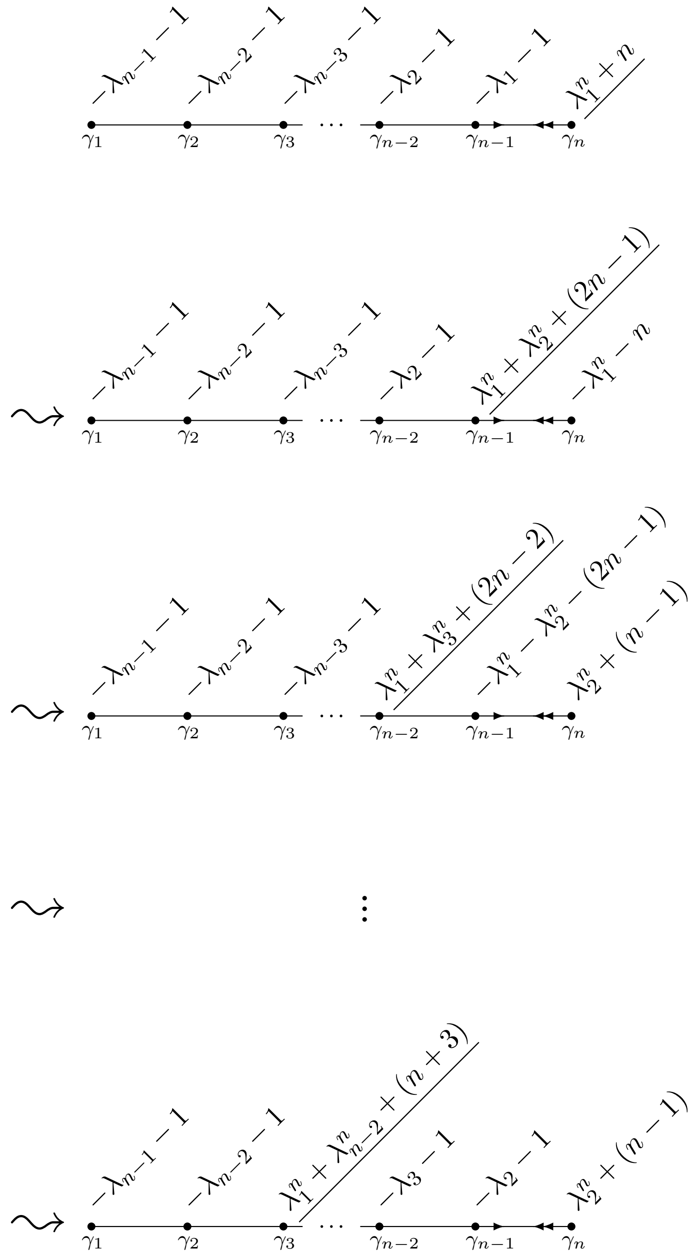
removing γ_n leaves a copy of A_{n-1}. In fact, starting with firing sequence $(\gamma_1; \gamma_2, \gamma_1; \gamma_3, \gamma_2, \gamma_1; \ldots; \gamma_n, \gamma_{n-1}, \ldots, \gamma_1)$ produces

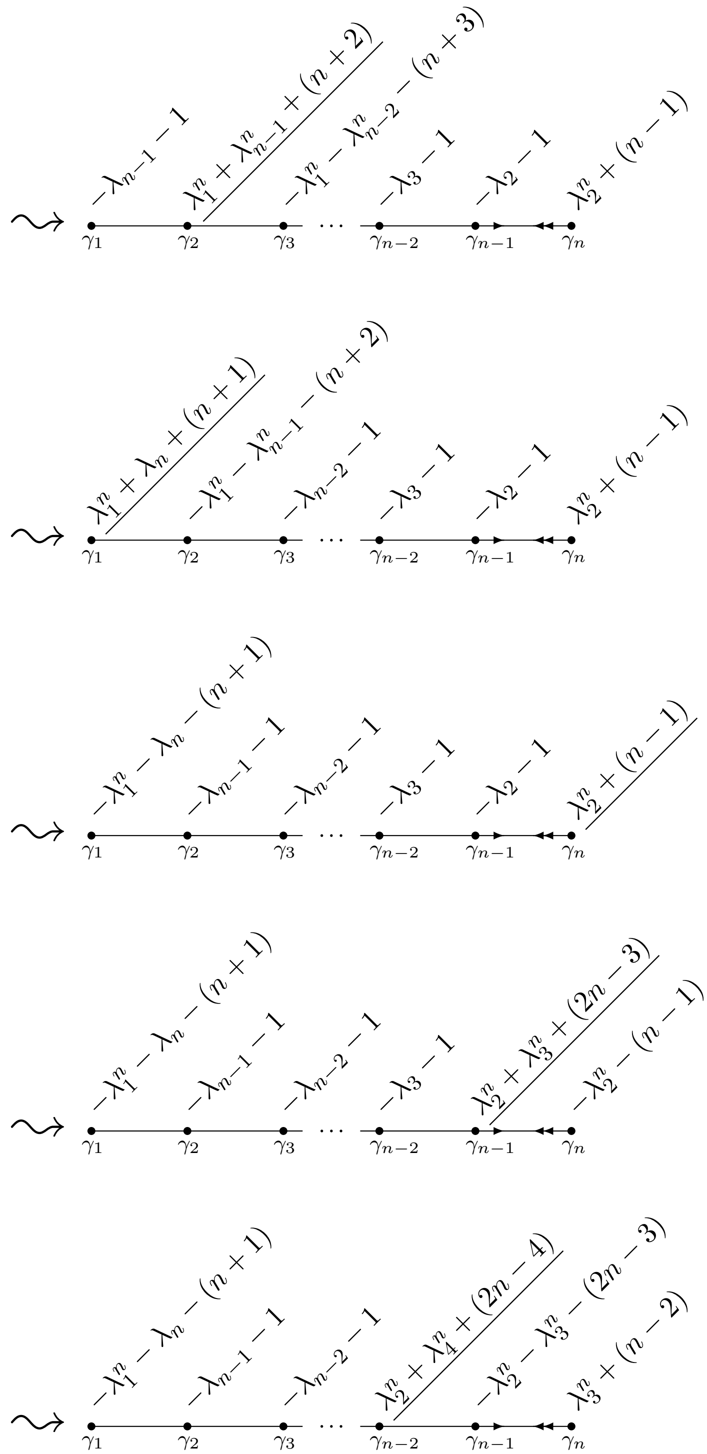


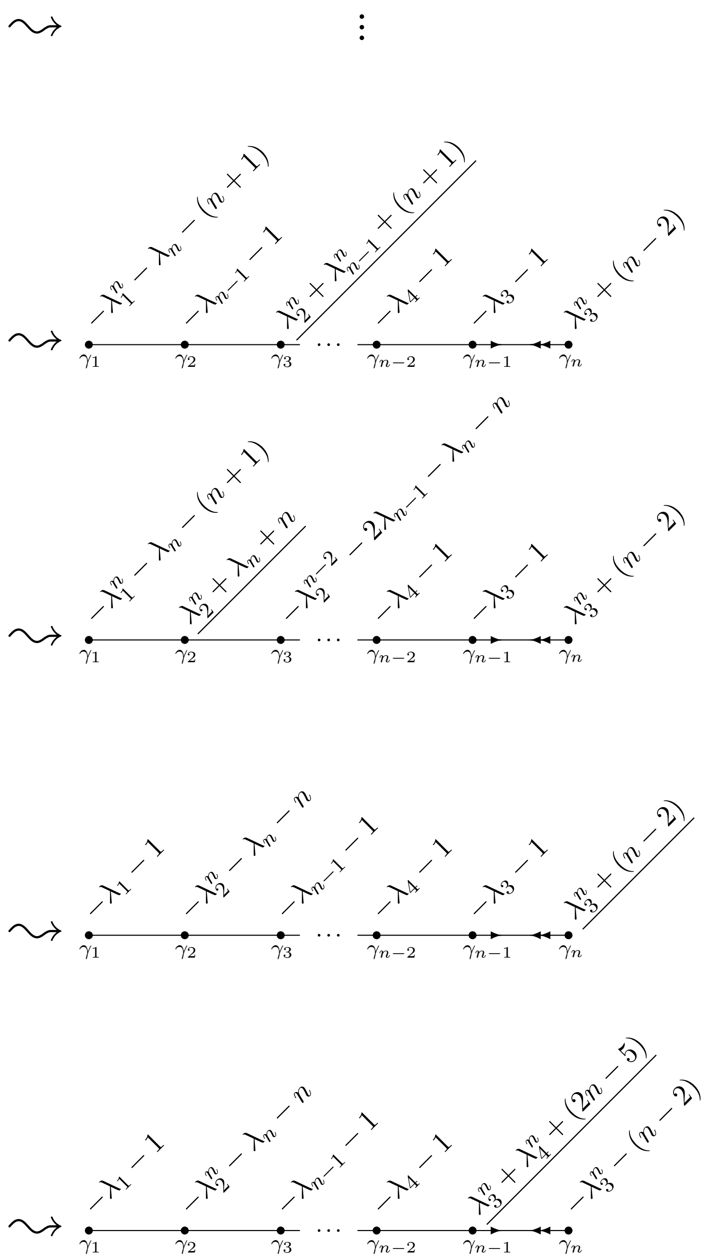
,

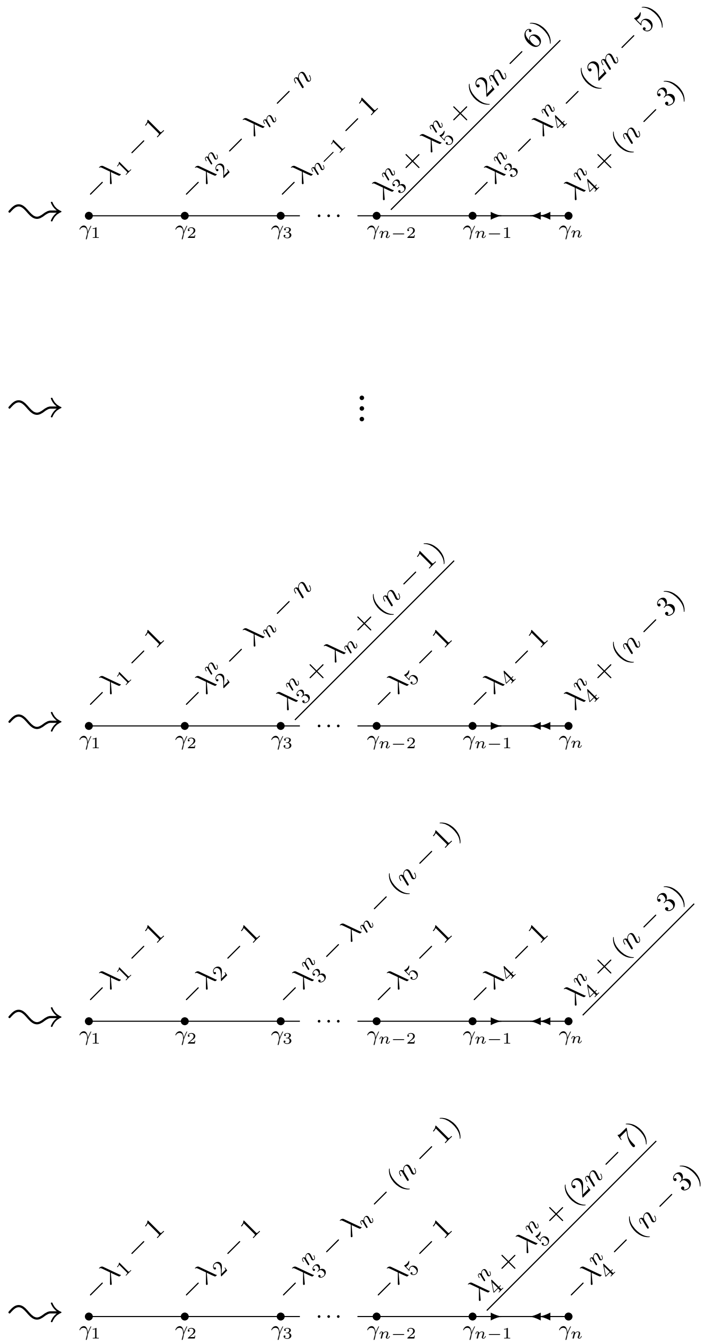
since the only time γ_{n-1} is fired it has population $\lambda_1^{n-1} + (n-1)$ and is added to $(\lambda_n + 1)$ at γ_n.

All that is left at this point is the second double product of equation A.2. Continuing gameplay using firing sequence $(\gamma_n, \gamma_{n-1}, \ldots, \gamma_1; \gamma_n, \gamma_{n-1}, \ldots, \gamma_2; \ldots; \gamma_n, \gamma_{n-1}; \gamma_n)$, we get

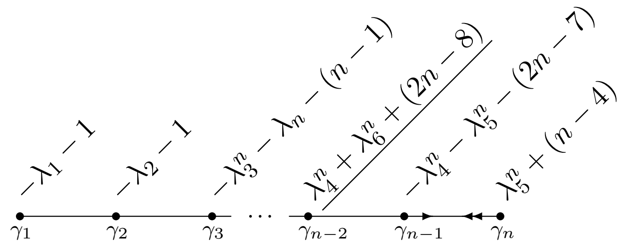

$-\lambda_{n-1}-1$
$-\lambda_{n-2}-1$
$-\lambda_{n-3}-1$
$-\lambda_2-1$
$-\lambda_1-1$
λ_1^n+n
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n
$\rightsquigarrow$
$-\lambda_{n-1}-1$
$-\lambda_{n-2}-1$
$-\lambda_{n-3}-1$
$-\lambda_2-1$
$\lambda_1^n+\lambda_2^n+(2n-1)$
$-\lambda_1^n-n$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n
$\rightsquigarrow$
$-\lambda_{n-1}-1$
$-\lambda_{n-2}-1$
$-\lambda_{n-3}-1$
$\lambda_1^n+\lambda_3^n+(2n-2)$
$-\lambda_1^n-\lambda_2^n-(2n-1)$
$\lambda_2^n+(n-1)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n
$\rightsquigarrow$
$\vdots$
$\rightsquigarrow$
$-\lambda_{n-1}-1$
$-\lambda_{n-2}-1$
$\lambda_1^n+\lambda_{n-2}^n+(n+3)$
$-\lambda_3-1$
$-\lambda_2-1$
$\lambda_2^n+(n-1)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n


$-\lambda_{n-1}-1$
$\lambda_1^n+\lambda_{n-1}^n+(n+2)$
$-\lambda_1^n-\lambda_{n-2}^n-(n+3)$
$-\lambda_3-1$
$-\lambda_2-1$
$\lambda_2^n+(n-1)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n
$\lambda_1^n+\lambda_n+(n+1)$
$-\lambda_1^n-\lambda_{n-1}^n-(n+2)$
$-\lambda_{n-2}-1$
$-\lambda_3-1$
$-\lambda_2-1$
$\lambda_2^n+(n-1)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n
$-\lambda_1^n-\lambda_n-(n+1)$
$-\lambda_{n-1}-1$
$-\lambda_{n-2}-1$
$-\lambda_3-1$
$-\lambda_2-1$
$\lambda_2^n+(n-1)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n
$-\lambda_1^n-\lambda_n-(n+1)$
$-\lambda_{n-1}-1$
$-\lambda_{n-2}-1$
$-\lambda_3-1$
$\lambda_2^n+\lambda_3^n+(2n-3)$
$-\lambda_2^n-(n-1)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n
$-\lambda_1^n-\lambda_n-(n+1)$
$-\lambda_{n-1}-1$
$-\lambda_{n-2}-1$
$\lambda_2^n+\lambda_4^n+(2n-4)$
$-\lambda_2^n-\lambda_3^n-(2n-3)$
$\lambda_3^n+(n-2)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n


$-\lambda_1^n - \lambda_n - (n+1)$
$-\lambda_{n-1} - 1$
$\lambda_2^n + \lambda_{n-1}^n + (n+1)$
$-\lambda_4 - 1$
$-\lambda_3 - 1$
$\lambda_3^n + (n-2)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n
$-\lambda_1^n - \lambda_n - (n+1)$
$\lambda_2^n + \lambda_n + n$
$-\lambda_2^{n-2} - 2\lambda_{n-1} - \lambda_n - n$
$-\lambda_4 - 1$
$-\lambda_3 - 1$
$\lambda_3^n + (n-2)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n
$-\lambda_1 - 1$
$-\lambda_2^n - \lambda_n - n$
$-\lambda_{n-1} - 1$
$-\lambda_4 - 1$
$-\lambda_3 - 1$
$\lambda_3^n + (n-2)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n
$-\lambda_1 - 1$
$-\lambda_2^n - \lambda_n - n$
$-\lambda_{n-1} - 1$
$-\lambda_4 - 1$
$\lambda_3^n + \lambda_4^n + (2n-5)$
$-\lambda_3^n - (n-2)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n

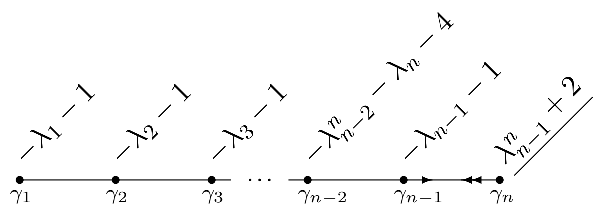

$-\lambda_1 - 1$
$-\lambda_2^n - \lambda_n - n$
$-\lambda_{n-1} - 1$
$\lambda_3^n + \lambda_5^n + (2n-6)$
$-\lambda_3^n - \lambda_4^n - (2n-5)$
$\lambda_4^n + (n-3)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n
$\vdots$
$-\lambda_1 - 1$
$-\lambda_2^n - \lambda_n - n$
$\lambda_3^n + \lambda_n + (n-1)$
$-\lambda_5 - 1$
$-\lambda_4 - 1$
$\lambda_4^n + (n-3)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n
$-\lambda_1 - 1$
$-\lambda_2 - 1$
$-\lambda_3^n - \lambda_n - (n-1)$
$-\lambda_5 - 1$
$-\lambda_4 - 1$
$\lambda_4^n + (n-3)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n
$-\lambda_1 - 1$
$-\lambda_2 - 1$
$-\lambda_3^n - \lambda_n - (n-1)$
$-\lambda_5 - 1$
$\lambda_4^n + \lambda_5^n + (2n-7)$
$-\lambda_4^n - (n-3)$
γ_1 γ_2 γ_3 $\cdots$ γ_{n-2} γ_{n-1} γ_n

$\rightsquigarrow$

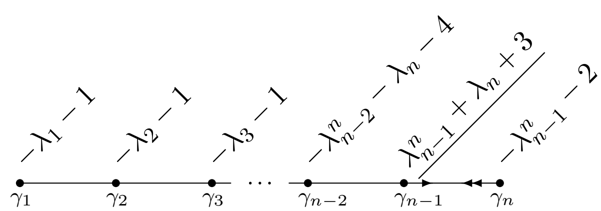


$\rightsquigarrow \qquad \vdots$

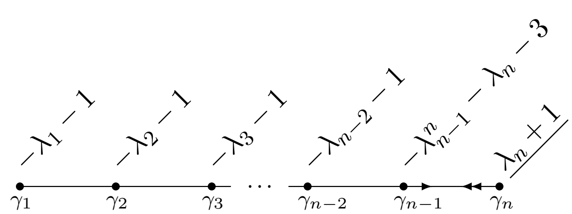
$\rightsquigarrow$

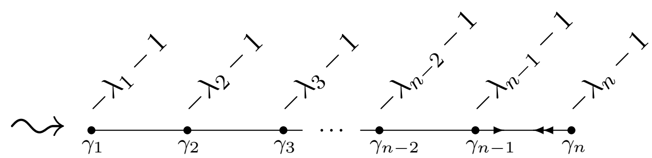


$\rightsquigarrow$



$\rightsquigarrow$





The underlined values in the above sequence do indeed match those in the second double product of equation A.2, and so the Quantum Dimension Polynomial Identity holds for type C_n.

$\square$

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