# Creating a Computational Tool to Simulate Vibration Control for Piezoelectric Devices

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## Abstract

A piezoelectric device is an elastic laminate having multiple bonded layers, at least one of which is made of a piezoelectric material. Piezoelectric material is a smart material (most notably Lead Zirconate Titanate) to develop electric displacement that is directly proportional to an applied mechanical stress. When a piezoelectric device is knocked off balance, it can result in vibrations adversely affecting sensitive components and therefore the quality of tasks performed. In many applications, it is not possible to wait until environmental influences dampen the vibration and bring it to a halt; moreover, several interferences usually overlap in time. The vibrations must therefore be insulated by the active piezoelectric components in order to dynamically decouple the structure from its surroundings and thus reduce the transmission of shocks and solid borne sound. Piezoelectric components have the ability to dampen vibrations particularly in the lower frequency range, either actively or passively. For this reason, first, vibrational dynamics on piezoelectric components have to be understood well. However, the existing mathematical models rely on oversimplified physics assumptions. This project aims to develop a reliable computational tool to simulate the control of vibrations on *a single piezoelectric bar*, described by a novel "partial differential equation" model. The existing models and their "unjustified" approximations in the literature are either heuristic or mathematically oversimplified. These models consider only the low-frequency vibrations. Our primary goal is to develop reproducible computational tools by an emerging stable approximation technique, so-called filtered semi-discrete or fully-discrete Finite Difference Method, which are proved to provide faster and reliable computation. Filtering in the approximation is necessary since the spurious vibrations, due to the blind application of the Finite Difference Method, provide a false stability result. The computational tool developed in this project is essential to provide new insights into the active controlling of piezoelectric devices involving piezoelectric components such as cardiac pacemakers or NASA/commercially-operated inflatable space antennas.

## Introduction

In mathematically modeling vibrations on a single piezoelectric beam, there are three major effects and their interrelations needed to be considered: mechanical, electrical, and magnetic. Mechanical effects are generally modeled through the Euler-Bernoulli small displacement assumptions. To include electrical and magnetic effects, there are mainly three approaches: electrostatic, quasi-static, and fully dynamic. Electrostatic and quasi-static approaches completely exclude magnetic effects and their coupling with electrical and mechanical effects. In this project, *even though the magnetic effects are relatively smaller*, the dynamic approach is used for to obtain the following system of partial differential equations (PDE):

$$(1)\quad \begin{cases} \rho u_{tt}-\alpha u_{xx}+\gamma\beta p_{xx}+\boldsymbol{\delta u_t}=0, & \\ \mu p_{tt}-\beta p_{xx}+\gamma\beta u_{xx}+\boldsymbol{\zeta p_t}=0, & (x,t)\in(0,L)\times(0,T_{final}) \\ u(0,t),p(0,t)=0, \alpha u_x(L,t)-\gamma\beta p_x(L,t)=0, -\beta p_x(L,t)+\gamma\beta u_x(L,t)=0, & t\in\mathbb{R}^+ \\ u(x,0)=u_0(x), p(x,0)=p_0(x), u_t(x,0)=u_1(x),\ p_t(x,0)=p_1(x), & x\in[0,L] \end{cases}$$

where $u(x,t)$ and $p(x,t)$ describe *longitudinal vibrations* and *total electrical charge* accumulated at the electrodes, and the positive coefficients $L, \rho, \alpha, \gamma, \beta, \mu, \delta, \zeta$ describe length, density, stiffness, piezoelectric, impermittivity, viscous and magnetic damping coefficients of the beam, respectively. In particular, the boldface terms are our controllers, see [1] for more details.

The solutions $u(x,t)$ and $p(x,t)$ of the model (1) are analytically shown to decay to zero exponentially by the distributed damping terms $\boldsymbol{\delta u_t}$ and $\boldsymbol{\zeta p_t}$. However, for implementing the control objective in practice, one has to approximate the model (1). There are various attempts in the literature to approximate controlled PDE, i.e. [3] and [5]. However, the well-known approximations fail to mimic the exponential decay of solutions $u(x,t)$ and $p(x,t)$. There are recent filtering techniques to improve these approximations such as **fully-discrete** or **filtered semi-discrete finite difference methods**. For a single PDE, i.e. a wave-type equation, each technique performs well [1]. However, for PDE models, where different dynamics are coupled, such as model (1), these techniques have not been shown to work well. The model (1) is a coupled wave system where each wave equation come with different wave propagation speeds $\sqrt{\frac{\alpha}{\rho}}$, $\sqrt{\frac{\beta}{\mu}}$. This is where the main challenge of this project is.

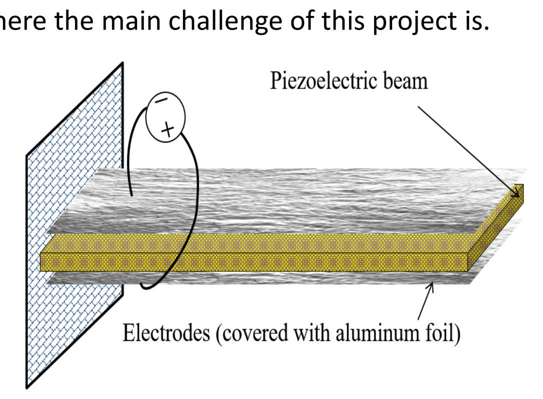



*Figure 1*: A piezoelectric beam covered with aluminum foil, clamped at one end and free at the other.

## Filtered Finite Differences

**I- Semi-discretized Finite Difference Scheme: Discretize with respect to $x$ only**

Consider the discretization of the interval $[0,L]$ with $dx=\frac{L}{N}$ and with the fictitious points $x_{-1}$ and $x_{N+1}$ as the following

$$x_{-1}<0=x_0<x_1<\cdots<x_i=i\cdot dx<\cdots<x_N<x_N=1<x_{N+1}$$

Notation: $\dot{u}_i=u_t(x_i,t)$ and $\ddot{u}_i=u_{tt}(x_i,t)$.

The following are the finite difference approximations of different order derivatives

$$u_x(x_i,t)\cong\frac{u(x_{i+1},t)-u(x_i,t)}{dx},\ u_{xx}(x_i,t)\cong\frac{u(x_{i+1},t)-2u(x_i,t)+u(x_{i-1},t)}{(dx)^2}.$$

Therefore, for $i=1,2,\cdots,N-1$, the system (1) is approximated as the following:

$$\text{(F-S-D)}\quad \begin{cases} \rho\ddot{u}_i-\alpha\frac{u_{i+1}-2u_i+u_{i-1}}{(dx)^2}-\alpha\boxed{\frac{\boldsymbol{u_{i+1}-2u_i+u_{i-1}}}{\mathbf{1}}}+\gamma\beta\frac{p_{i+1}-2p_i+p_{i-1}}{(dx)^2}+\boldsymbol{\delta\dot{u}_i}=0 \\ \mu\ddot{p}_i-\beta\frac{p_{i+1}-2p_i+p_{i-1}}{(dx)^2}-\beta\boxed{\frac{\boldsymbol{p_{i+1}-2p_i+p_{i-1}}}{\mathbf{1}}}+\gamma\beta\frac{u_{i+1}-2u_i+u_{i-1}}{(dx)^2}+\boldsymbol{\zeta\dot{p}_i}=0 \end{cases}$$

Boundary conditions: $u_0=p_0=0,\quad \alpha\frac{u_{N+1}-u_N}{dx}-\gamma\beta\frac{p_{N+1}-p_N}{dx}=0$

$$-\beta\frac{p_{N+1}-p_N}{dx}=+\gamma\beta\frac{u_{N+1}-u_N}{dx}=0$$

Initial conditions: $u_i(0)=u_0(x_i),\quad p_i(0)=p_0(x_i),\quad \dot{u}_i(0)=u_1(x_i),\quad \dot{p}_i(0)=p_1(x_i)$

The boxed terms are called numerical viscosity terms (for filtering) added to each equation. As the equations are multiplied by $(dx)^2$, the boxed terms tend to zero $as\ dx\to 0$. This way, the spurious high-frequency vibrations, are prevented to destabilize the system.

**II- Fully-discretized Finite Difference Scheme: Discretize with respect to both $x$ and $t$.**

In addition to the space discretization of first method, consider the time discretization of the interval $[0,T_f]$ with $dx=\frac{1}{M}$ as the following:

$$0=t_0<t_1<\cdots<t_N=T_f.$$

The following are the finite difference approximations of different order derivatives

$$u_t(x_i,t_j)\cong\frac{u_i^{j+1}-u_i^j}{dt},\quad u_x(x_i,t_j)\cong\frac{u_{i+1}^j-u_i^j}{dx},$$

$$u_{tt}(x_i,t_j)\cong\frac{u_i^{j+1}-2u_i^j+u_i^{j-1}}{(dt)^2},\qquad u_{xx}(x_i,t_j)\cong\frac{u_{i+1}^j-2u_i^j+u_{i-1}^j}{(dx)^2}$$

## Filtered Finite Differences (Cont.)

Therefore, the system (1) is approximated as the following

$$\text{(F-D)}\quad \begin{cases} \rho\frac{u_i^{j+1}-2u_i^j+u_i^{j-1}}{dt^2}-\alpha\frac{u_{i+1}^j-2u_i^j+u_{i-1}^j}{dx^2}+\gamma\beta\frac{p_{i+1}^j-2p_i^j+p_{i-1}^j}{dx^2}+\boxed{\boldsymbol{\delta}\frac{\boldsymbol{u_i^{j+1}-u_i^{j-1}}}{\mathbf{2dt}}}=0, \\ \mu\frac{p_i^{j+1}-2p_i^j+p_i^{j-1}}{dt^2}-\beta\frac{p_{i+1}^j-2p_i^j+p_{i-1}^j}{dx^2}+\gamma\beta\frac{u_{i+1}^j-2u_i^j+u_{i-1}^j}{dx^2}+\boxed{\boldsymbol{\zeta}\frac{\boldsymbol{p_i^{j+1}-p_i^{j-1}}}{\mathbf{2dt}}}=0, \end{cases}$$

Boundary conditions: $u_0^j=p_0^j=0,\quad \alpha\frac{u_{N+1}^j-u_N^j}{dx}-\gamma\beta\frac{p_{N+1}^j-p_N^j}{dx}-=0\ ,-\beta\frac{p_{N+1}^j-p_N^j}{dx}+\gamma\beta\frac{u_{N+1}^j-u_N^j}{dx}=0,$

Initial conditions: $u_i^0(0)=u_0(x_i),\quad p_i^0(0)=p_0(x_i),\quad \frac{u_i^1-u_i^0}{dt}=u_1(x_i),\quad \frac{p_i^1-p_i^0}{dt}=p_1(x_i).$

## Results

We consider a sample piezoelectric beam with height L = 1m, thickness h = 0.01m and material constants $\rho$ = 7600 kg/m$^3$, $\alpha_1$ = $7.6\times10^7$ N/m$^2$, $\gamma$ = $3\times10^{-4}$ C/m$^2$ , $\beta$ = $1.9\times10^{-5}$ m/F, and $\mu$ = $1.2\times10^{-3}$N/A$^2$. We took N=17 and M = 7001 for the fully-discretized simulations, such that the Courant number was less than 1, and N=80 for the semi-discretized simulations, and set the controller gains for the corresponding mechanical $\delta$ and electrical damping terms and $\zeta$. These gains are in the range of [10^5,10^7]. We simulated both low and high-frequency initial conditions using both the fully-discretized and semi-discretized finite difference schemes. For all simulations, $T_{final}$ = 3 sec. The semi-discrete scheme fails to provide good numerical results for the high-frequency initial conditions due to the sensitivity of the code to initial conditions and parameters.

### Low-Frequency Initial Conditions

We consider

$$u_0(x)=5\times10^{-3}\big(Sin(\pi x)+Sin(3\pi x)\big),\qquad p_0(x)=2\times10^6\ Sin(5\pi x),\qquad u_1(x)=p_1(x)=0$$

**Fully-Discretized (F-D)** **Filtered Semi-Discretized (F-S-D)**

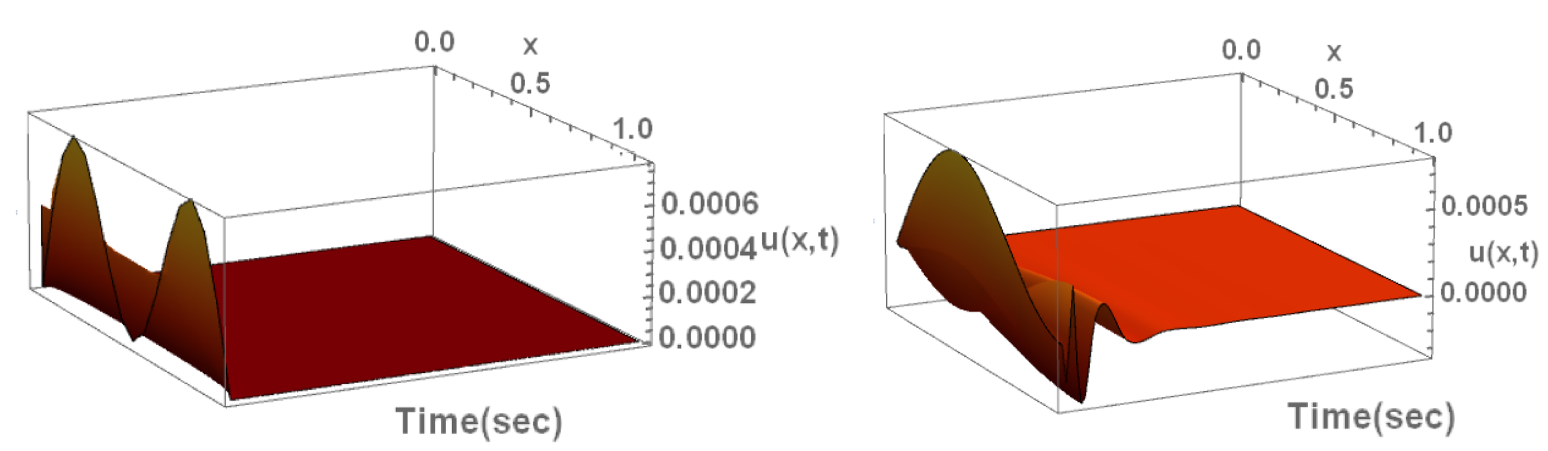



*Figure 2*: Graphs of the longitudinal vibration of the piezoelectric beam over 3 seconds with low-frequency initial conditions for the the F-D (left) and the F-S-D (right) finite difference schemes.

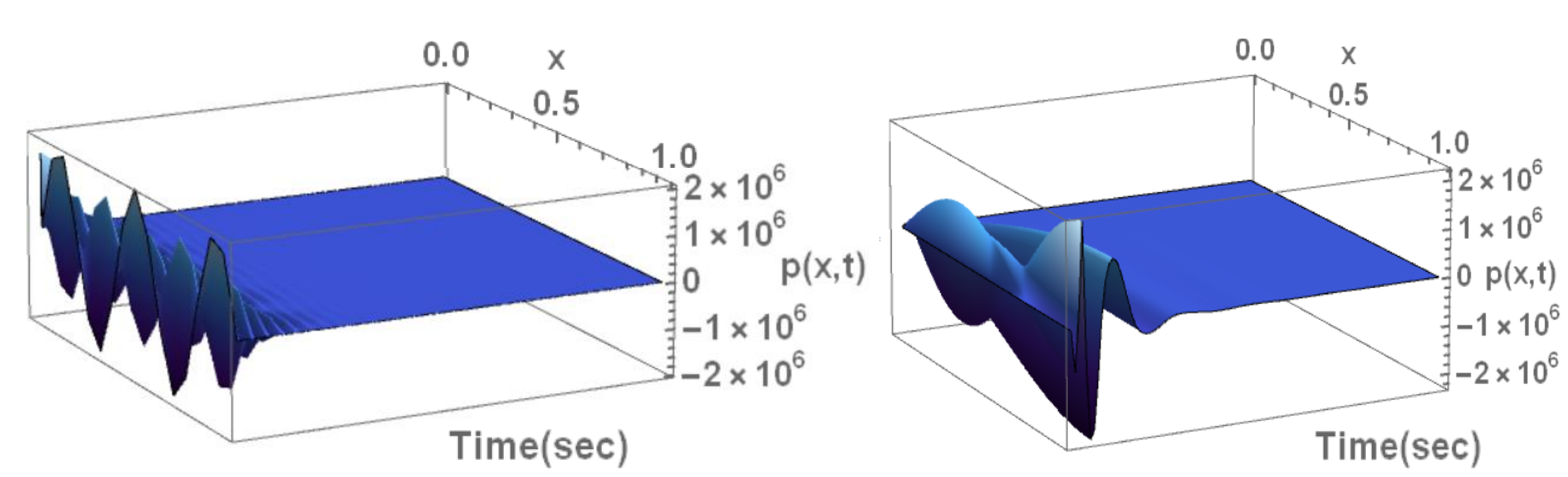



*Figure 3*: Graphs of the total charges on the electrodes of the piezoelectric beam over 3 seconds with low-frequency initial conditions the F-D (left) and the F-S-D (right) finite difference schemes.

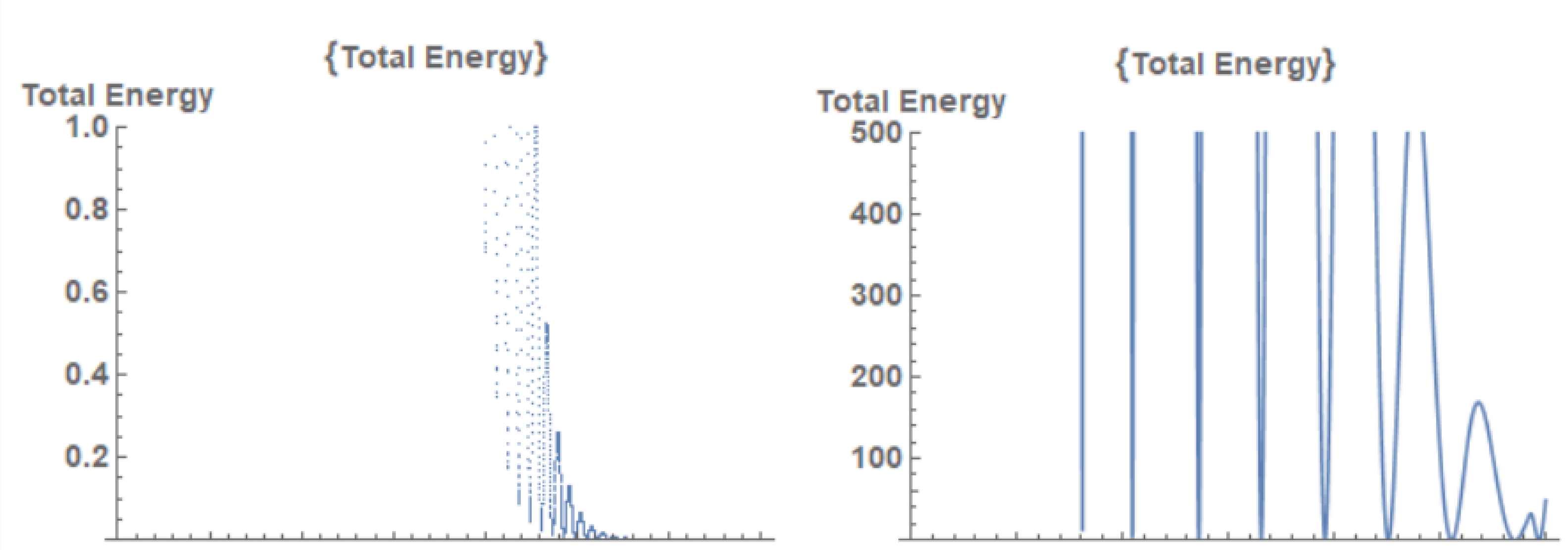



*Figure 4*: Graphs of the total energy (mechanical + electrical + magnetic) of the piezoelectric beam over 3 seconds with low-frequency initial conditions for the F-D (left) and the F-S-D (right) finite difference schemes.

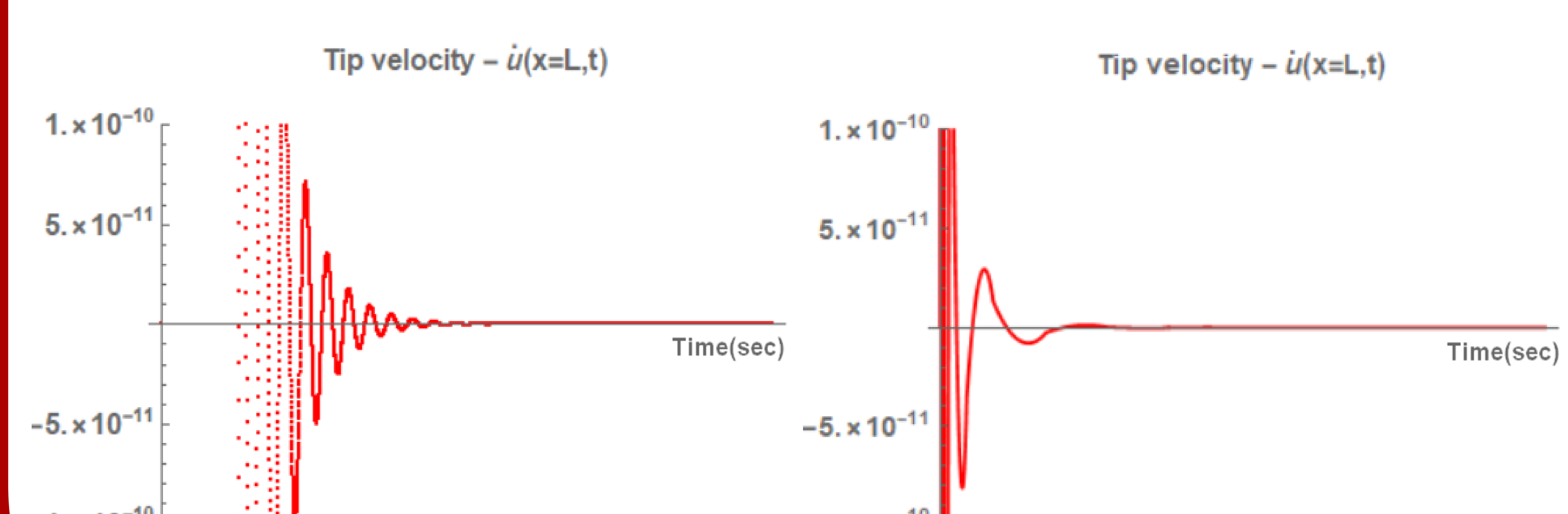



*Figure 5*: Graphs of the tip velocity of the piezoelectric beam over 3 seconds with low-frequency initial conditions for the F-D (left) and the F-S-D (right) finite difference schemes.

## Results (Cont.)

### High-Frequency Initial Conditions (only for the F-D case)

We consider the initial conditions

$$u_0(x)=1\times10^{-4}Sin(13\pi x),\qquad p_0(x)=3\times10^6Sin(15\pi x),\qquad u_1(x)=p_1(x)=0$$

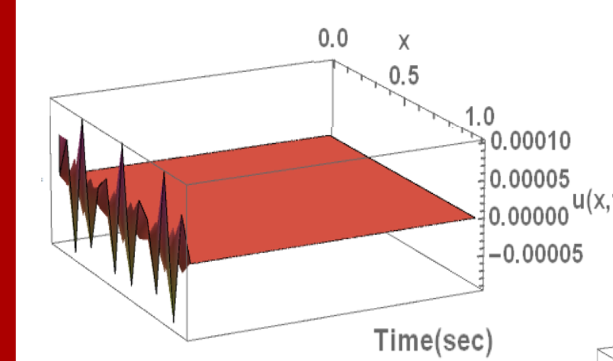
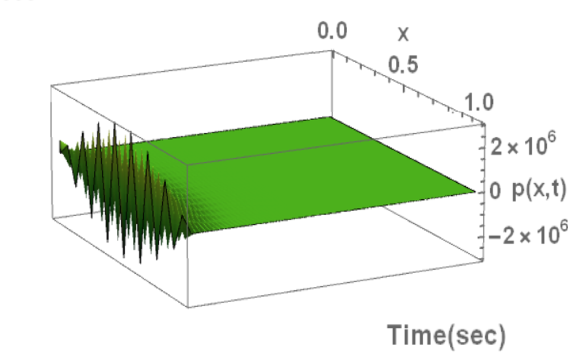
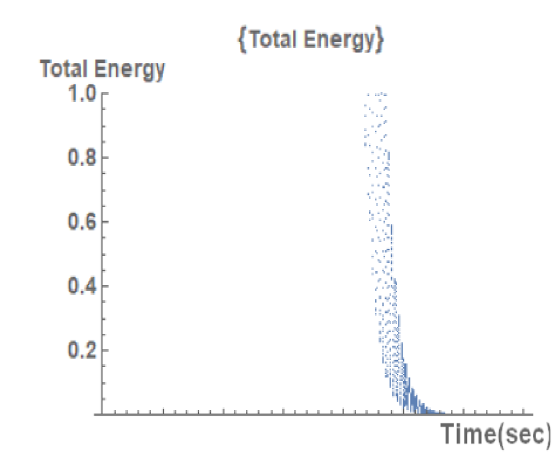



*Figure 6*: Graphs of the longitudinal vibrations (left) and total electrical charges (below) of the piezoelectric beam over 3 seconds with high-frequency initial conditions for the F-D finite difference schemes.



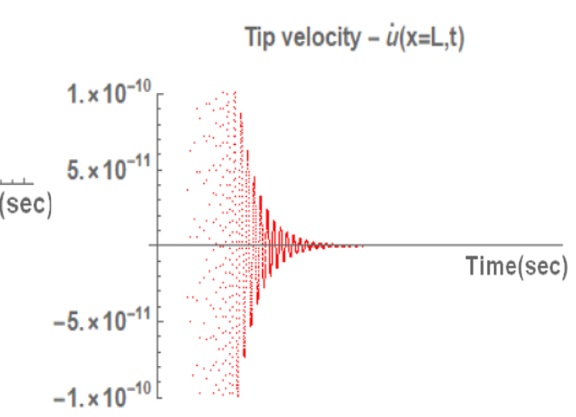




*Figure 7*: Graphs of the total energy (above) and tip velocity (right) of the piezoelectric beam over 3 seconds with high-frequency initial conditions for the F-D finite difference schemes.

## Discussion and Open Problems

- The results in the low-frequency range mimic the analytical results in both cases. Both mechanical and electrical vibrations decay to zero in time. The fast decay of the energy and tip-velocity show that the fully-discrete scheme can be perfectly used in practical applications.
- However, as the frequency range increases the filtered semi-discrete approximations fail to perform well. It is observed the approximation is sensitive to initial conditions and control gain parameters. This is still something that has to be addressed.
- In general, the fully-discrete approximations perform better in simulations since they are just based on iterations. The filtered semi-discrete approximations are more accurate even though they are computationally expensive.
- The simulations can also be performed for different combinations of initial configurations of the beam in both approximations.
- In the model, there are two damping controllers $\boldsymbol{\delta u_t}$ and $\boldsymbol{\zeta p_t}$. Removing one of them fails to provide exponential decay of solutions even though analytical techniques show that one damping controller added only to either one of the equations is good enough for exponential decay.

### Open Problems and research in progress:

1) Control by a single damping term: Analytic results have proven that a single damping term $\boldsymbol{\delta u_t}$ in the $u$-equation of (1) should be enough to damp both electrical and mechanical vibrations. In the fully-discretized scheme, two damping terms perfectly stabilize low and high-frequency vibrations whereas in the semi-discrete scheme the same terms only stabilize the $u$-equation. We are currently working towards eliminating the damping term $\boldsymbol{\zeta p_t}$ in the $p$-equation.

2) Boundary dampers (right boundary): Our current model uses distributed dampers. Boundary dampers, while relatively weaker, are more cost effective. Proving that two boundary dampers, added to the boundary conditions, provide good damping results and this is proved analytically. Developing a filtered finite difference scheme is an ongoing project.

3) More advanced designs: In the future, we plan to simulate piezoelectric materials distributed to layers of a laminate in an alternating way, allowing the piezoelectric material to be used as an active controller such as [2] and [4]. The main challenge here is the coupling of wave dynamics with a fourth order partial differential equations describing bending of the laminate. As far as the current literature goes, there are not any results reported.

## Literature Cited

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## Acknowledgements

We would like to thank the WKU Research Office for selecting our project for a FUSE Grant.