2018

$W^{1,p}$ Regularity of Eigenfunctions for the Mixed Problem with Nonhomogeneous Neumann Data

Kohei Miyazaki
Murray State University

$W^{1,p}$ Regularity of Eigenfunctions for the Mixed Problem with Nonhomogeneous Neumann Data

A Thesis

Presented to

the Faculty of the Department of Mathematics and Statistics

Murray State University

Murray, Kentucky

In Partial Fulfillment

of the Requirements for the Degree

of Master of Science

by

Kohei Miyazaki

May 2018

Acknowledgements

I owe a great amount of gratitude to my advisor, Dr. Justin Taylor, who has always been patient and provided me with invaluable mentorship throughout my time of writing this thesis. I am truly grateful to call him my advisor.

I also owe my deepest thanks to my parents. I could not have written this thesis without their support.

Abstract

We consider an eigenvalue problem with a mixed boundary condition, where a second-order differential operator is given in divergence form and satisfies a uniform ellipticity condition. We show that if a function u in the Sobolev space $W_D^{1,2}$ is a weak solution to the eigenvalue problem, then u also belongs to $W_D^{1,p}$ for some $p > 2$. To do so, we show a reverse Hölder inequality for the gradient of u. The decomposition of the boundary is assumed to be such that we get both Poincaré and Sobolev-type inequalities up to the boundary.

Contents

Chapter 1

Introduction

Throughout this paper, Ω denotes some open subset of $\mathbb{R}^n$. All functions are real-valued.

1.1 Motivation

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$. For a fixed integer $k \geq 1$, the general k^{th} order partial differential equation (PDE) can be written as

$$F(D^k u(x), D^{k-1}u(x), \ldots, Du(x), u(x), x) = 0, \tag{1.1}$$

where $x \in \Omega$, $u : \Omega \to \mathbb{R}$ is the unknown function of x, $D^k u(x)$ denotes a k^{th} partial derivative of u, and F is a given function that relates x, u and certain partial derivatives of u. We say that $u : \Omega \to \mathbb{R}$ is a *classical solution* or a *solution in the classical sense* to the PDE (1.1) if u is k-times continuously differentiable on Ω, and u and its partial derivatives satisfy (1.1). PDEs modeling real world phenomena often exhibit some singularities or other non-smooth behaviors, and in general we cannot expect such PDEs to have a classical solution. Moreover, even when a PDE does have a classical solution, it can be extremely difficult to prove the existence of a classical solution directly. To overcome these issues, we extend the notion of solutions and discuss *weak* (or *generalized*) solutions, which are functions that may be less smooth than classical solutions, but satisfy *some* conditions prescribed by the PDE (the precise definition of weak solutions will be given in chapter 3).

Now, if u is a classical solution to a k^{th} order PDE, then we know u has some degree of regularity (i.e. u is at least k-times continuously differentiable). On the other hand, weak solutions generally do not possess as much regularity as classical solutions, and determination of regularity of weak solutions is a nontrivial task. In this paper, we study regularity of weak solutions to an eigenvalue problem with a mixed boundary condition, which assigns a homogeneous Dirichlet boundary condition and a nonhomogeneous Neumann boundary condition on different parts of the boundary of the domain. We seek solutions from the Sobolev space $W_D^{1,2}(\Omega)$, which comprises those functions in $W^{1,2}(\Omega)$ that vanish near the Dirichlet part of the boundary, and show that eigenfunctions possess slightly higher regularity than the typical function in $W_D^{1,2}(\Omega)$. This result is vital and applicable to a number of problems in PDE theory, including the study of eigenvalue convergence on perturbed domains.

1.2 Basic Definitions and Notations

Euclidean Space

We denote an n-tuple by x, rather than $\mathbf{x}$ or $\vec{x}$. The Euclidean norm of $x = (x_1, x_2, \dots, x_n)$ is denoted by

$$|x| = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{\frac{1}{2}}.$$

We write x^0 to emphasize that the point $x^0 \in \mathbb{R}^n$ is fixed. We write dx for the differential when integrating with respect to n-dimensional Lebesgue measure and $d\sigma(x)$ when integrating with respect to $(n-1)$-dimensional Lebesgue measure. If

$E \subset \mathbb{R}^n$ is an n-dimensional set, then we write $|E|$ to denote the n-dimensional Lebesgue measure of E. If $G \subset \mathbb{R}^n$ is an $(n-1)$-dimensional set, then we write $\sigma(G)$ to denote the $(n-1)$-dimensional Lebesgue measure of G. The closure of E is denoted by $\overline{E}$, the interior by E°, and the boundary by $\partial E = \overline{E} \backslash E^{\circ}$. We say that an open set G is compactly contained in E if $\overline{G}$ is a bounded subset of E.

Let $x^0 \in \mathbb{R}^n$ and $r > 0$. The open ball centered at x^0 with radius r is denoted by

$$B_r(x^0) := \{x \in \mathbb{R}^n : |x - x^0| < r\}.$$

For $x^0 \in \Omega$, we define

$$\Omega_r(x^0) := \begin{cases} B_r(x^0) & \text{if} \quad B_r(x^0) \cap \partial\Omega = \emptyset \\ B_{2r}(\tilde{x}) \cap \Omega & \text{if} \quad B_r(x^0) \cap \partial\Omega \neq \emptyset \end{cases}$$

and

$$\Delta_r(x^0) := \begin{cases} \emptyset & \text{if} \quad B_r(x^0) \cap \partial\Omega = \emptyset \\ B_{2r}(\tilde{x}) \cap \partial\Omega & \text{if} \quad B_r(x^0) \cap \partial\Omega \neq \emptyset \end{cases},$$

where $\tilde{x}$ is the point on $B_r(x^0) \cap \partial\Omega$ that is closest to x^0.

The n-dimensional Lebesgue measure of $B_r(x^0)$ is given by

$$|B_r(x^0)| = \frac{\pi^{\frac{n}{2}} r^n}{\Gamma(\frac{n}{2} + 1)},$$

where Γ is the gamma function defined by

$$\Gamma(z) := \int_0^{\infty} x^{z-1} e^{-x} \, dx$$

for any $z > 0$. Note that if $\Delta_r(x^0) \neq \emptyset$, then we can write $|B_r(x^0)| = Cr\sigma(\Delta_r(x^0))$ where C is some constant only depending on the dimension.

Notation for Derivatives

If u is a function of several variables, we write u_{x_i} to denote $\frac{\partial u}{\partial x_i}$, the partial derivative of u with respect to x_i. An n-tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, where each component α_i is a nonnegative integer, is called a *multi-index*. The order of α, denoted by $|\alpha|$, is defined to be $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Given a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, we define a differential operator D^α by

$$D^\alpha := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

The *order* of D^α is defined to be $|\alpha|$. Note that $D^\alpha u = u$ if $|\alpha| = 0$. If $k \in \mathbb{N}$, we also write $D^k u$ to denote $D^\alpha u$ for any α of order k. The gradient of u is denoted by $\nabla u := (u_{x_1}, u_{x_2}, \ldots, u_{x_n})$.

Spaces of Continuous Functions and Regularity of the Boundary

For $k \in \mathbb{N}$, $C^k(\Omega)$ denotes the space of all functions $u : \Omega \to \mathbb{R}$ such that for each α satisfying $|\alpha| \leq k$, $D^\alpha u$ exists and is continuous on Ω. We define $C^\infty(\Omega) := \bigcap_{k=0}^\infty C^k(\Omega)$. Functions in $C^\infty(\Omega)$ are said to be *smooth*. If u is a function defined on Ω, then the *support* of u, denoted by $\operatorname{supp}(u)$, is defined by

$$\operatorname{supp}(u) := \overline{\{x \in \Omega : u(x) \neq 0\}}.$$

We write $C_c^\infty(\Omega)$ to denote the space of all smooth functions $u : \Omega \to \mathbb{R}$ whose support is compactly contained in Ω. Thus, functions in $C_c^\infty(\Omega)$ are infinitely differentiable and vanish outside some bounded subset of Ω. If $\Omega \subset \mathbb{R}^n$ is bounded, then the space $C^k(\overline{\Omega})$ is defined to be the class of all functions $u : \Omega \to \mathbb{R}$ such that for each α with $|\alpha| \leq k$, $D^\alpha u$ exists and is uniformly continuous on Ω. One can show that if $u \in C^k(\overline{\Omega})$, then for each α with $|\alpha| \leq k$, $D^\alpha u$ can be extended continuously to $\overline{\Omega}$, where such extension is unique.

Let $k \in \mathbb{N}$. We say that the boundary $\partial\Omega$ is C^k if for each point $x^0 \in \partial\Omega$ there exists $r > 0$ and a function $\gamma \in C^k(\mathbb{R}^{n-1})$ such that, after relabeling and reorienting the coordinates axes if necessary, we have

$$\Omega \cap B_r(x^0) = \{x \in B_r(x^0) : x^n > \gamma(x_1, x_2, \ldots, x_{n-1})\}.$$

We interpret a C^k-boundary as being locally the graph of a C^k-function after a change of coordinates if necessary. We say that $\partial\Omega$ is C^∞ if $\partial\Omega$ is C^k for all $k \in \mathbb{N}$.

Lebesgue Spaces

Let E be a measurable subset of $\mathbb{R}^n$ and u be a measurable function on E. The *essential supremum of* u *on* E, denoted by $\operatorname*{ess\,sup}_{x\in E} u(x)$, is defined to be

$$\operatorname*{ess\,sup}_{x\in E} u(x) := \inf\{\alpha \in \mathbb{R} : |\{x \in E : u(x) > \alpha\}| = 0\}.$$

The essential supremum of u on E may be thought of as the supremum of u on E disregarding the values that u takes on a subset Z of E with $|Z| = 0$.

Example 1.1. Consider the *Dirichlet function* defined by

$$u(x) := \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{if} \quad x \notin \mathbb{Q}\ . \end{cases}$$

Then $\operatorname*{ess\,sup}_{x\in E} u(x) = 0$ since $|\mathbb{Q}| = 0$. Note that $\operatorname*{ess\,sup}_{x\in E} u(x) = \operatorname*{ess\,sup}_{x\in E} v(x)$ whenever $u = v$ a.e. on E.

We say that u is *essentially bounded* on E if

$$\operatorname*{ess\,sup}_{x\in E} |u(x)| < \infty.$$

Let E be a measurable subset of $\mathbb{R}^n$ and let $1 \le p < \infty$. The *Lebesgue space*, denoted by $L^p(E)$, is defined to be the class of all measurable functions $u : E \to \mathbb{R}$ such that

$$\int_E |u|^p \, dx < \infty. \tag{1.2}$$

If $u \in L^p(E)$, then u is said to be L^p*-integrable* on E. The space $L^\infty(E)$ is defined to be the class of all measurable functions that are essentially bounded on E.

Recall that a vector space X is called a *normed space* over $\mathbb{R}$ if there exists a function $\|\cdot\| : X \to \mathbb{R}$, called a *norm*, such that

(i) $\|x\| = 0$ if and only if x is the zero element of X,

(ii) $\|\lambda u\| = |\lambda|\|u\|$ for $x \in X$ and $\lambda \in \mathbb{R}$,

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for $x, y \in X$.

A normed space X is said to be *complete* with respect to the metric induced by its norm if every Cauchy sequence in X converges to an element of X; that is, if $\{x_k\}$ is a sequence in X such that $\|x_k - x_l\| \to 0$ as $k, l \to \infty$, then there exists an element $x \in X$ such that $\|x_k - x\| \to 0$ as $k \to \infty$. A normed space that is complete with respect to the metric induced by its norm is called a *Banach space*.

One can readily verify that for any $1 \leq p \leq \infty$, the space $L^p(E)$ is a Banach space with respect to the norm given by

$$\|u\|_{L^p(E)} := \begin{cases} \left(\int_E |u(x)|^p \, dx \right)^{\frac{1}{p}} & \text{if} \quad 1 \leq p < \infty \\ \operatorname*{ess\,sup}_{x \in E} |u(x)| & \text{if} \quad p = \infty. \end{cases}$$

The triangle inequality for this norm is called *Minkowski's inequality* (see Theorem A.2). Note that since the value of integrals or the value of essential supremums are not affected when we change the value of functions on a set of measure zero, for any $1 \leq p \leq \infty$, $\|u\|_{L^p(E)} = \|v\|_{L^p(E)}$ whenever $u = v$ a.e. on E. This means that elements of the spaces $L^p(E)$ are in fact equivalence classes of measurable functions, where the equivalence relation $\sim$ is defined by

$$u \sim v \quad \text{if and only if} \quad u = v \quad \text{a.e. on } E.$$

Nevertheless, we often ignore this distinction and identify a function with its equivalence class for convenience. Observe that for any $1 \leq p < q < \infty$, the Hölder's inequality implies

$$L^q(E) \subset L^p(E), \tag{1.3}$$

whenever $|E| < \infty$.

Suppose $u : \Omega \to \mathbb{R}$ is measurable and let $1 \leq p \leq \infty$. If $u \in L^p(K)$ for every compact subset K of Ω, then u is said to be *locally L^p-integrable* on Ω. The class of all locally L^p-integrable functions is denoted by $L^p_{\text{loc}}(\Omega)$. Note that for any $1 \leq p \leq \infty$, the space $L^p_{\text{loc}}(\Omega)$ contains a larger class of functions than $L^p(\Omega)$.

Chapter 2

Sobolev Spaces

Sobolev spaces are named after a Russian mathematician Sergei Sobolev. These spaces are vector subspaces of various L^p spaces, and consist of functions whose partial derivatives also belong to L^p spaces. Their significance lies on the fact that weak solutions to many PDEs are naturally found in Sobolev spaces rather than the space of smooth functions. The following is a quote by Gaetano Fichera, taken from his “Analytic Problems of Hereditary Phenomena” (1977), which can be found in Graffi [6].

> These spaces, at least in the particular case $p = 2$, were known since the very beginning of this century, to the Italian mathematician Beppo Levi and Guido Fubini who investigated the Dirichlet minimum principle for elliptic equations. Later on many mathematicians have used these spaces in their work. Some French mathematicians, at the beginning of the fifties, decided to invent a name for such spaces as, very often, French mathematicians like to do. They proposed the name Beppo Levi spaces. Although this name is not very exciting in the Italian language and it sounds because of the name “Beppo”, somewhat peasant, the outcome in French must be gorgeous since the special French pronunciation of the names makes it to sound very impressive. Unfortunately,

this choice was deeply disliked by Beppo Levi, who at that time was still alive, and - as many elderly people - was strongly against the modern way of viewing mathematics. In a review of a paper of an Italian mathematician, who, imitating the Frenchmen, had written something on "Beppo Levi spaces", he practically said that he did not want to leave his name mixed up with this kind of things. Thus the name had to be changed. A good choice was to name the spaces after S. L. Sobolev. Sobolev did not object and the name Sobolev spaces is nowadays universally accepted.

2.1 Weak Derivatives

Before giving the definition of Sobolev spaces, we need to discuss what it means for a function to be weakly differentiable. The notion of weak derivatives is important because there are many functions that are deemed to satisfy the conditions given by PDEs, but not differentiable in the ordinary sense (recall that in order for a function to be a classical solution to a k^{th} order PDE, the function at least has to be k-times differentiable in the ordinary sense). By introducing a weaker notion of derivatives, such functions could become "weak solutions" to PDEs.

If $\partial\Omega$ is C^1, define a vector field $\boldsymbol{\nu} : \partial\Omega \to \mathbb{R}^n$ by

$$\boldsymbol{\nu}(x) := (\nu_1, \nu_2, \ldots, \nu_n),$$

where $(\nu_1, \nu_2, \ldots, \nu_n)$ denotes the outward unit normal at $x \in \partial\Omega$. The following integration by parts formula is a corollary of the Divergence Theorem.

Corollary 2.1 (Integration by parts formula)**.** *Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^1-boundary. If $u, v \in C^1(\overline{\Omega})$, then*

$$\int_\Omega u v_{x_i}\,dx = -\int_\Omega u_{x_i} v\,dx + \int_{\partial\Omega} u v \nu_i\,d\sigma$$

for $i = 1, 2, \ldots, n$, where ν_i is the i^{th} component of the outward unit normal at $x \in \partial\Omega$.

We now define weak derivatives of a function. To motivate the definition, first suppose $f \in C^1(\Omega)$. For any smooth function $\varphi \in C_c^\infty(\Omega)$, it follows from Corollary 2.1 that

$$\int_\Omega f\varphi_{x_i}\,dx = -\int_\Omega f_{x_i}\varphi\,dx \tag{2.1}$$

for $i = 1, 2, \ldots, n$. There is no boundary term since φ vanishes near $\partial\Omega$. Now let us see if there is any weaker assumption than f being in $C^1(\Omega)$ under which (2.1) would make sense. First, observe that if f is only locally integrable (and not necessarily differentiable) on Ω, then the integral on the left side of (2.1) will still be finite. To see this, suppose $f \in L^1_{\text{loc}}(\Omega)$ and $\varphi \in C_c^\infty(\Omega)$. Let K_0 denote the compact support of φ in Ω. Then, clearly K_0 contains the compact support of φ_{x_i} in Ω for each $i = 1, 2, \ldots, n$. Hence,

$$\int_\Omega f\varphi_{x_i}\,dx = \int_{K_0} f\varphi_{x_i}\,dx \leq \int_{K_0} fM\,dx \leq M\int_{K_0}|f|\,dx < \infty,$$

where $M = \sup_{x\in K_0}\varphi_{x_i}(x)$, which is finite since continuous functions are bounded on a compact set. The expression f_{x_i} on the right side of (2.1) has no obvious meaning if f is not differentiable on Ω. In this case, we define a function g, which may not be the derivative of f in the ordinary sense, to be the *weak derivative* of f with respect to x_i if g plays the role of f_{x_i} in (2.1). The precise definition is as follows:

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be open, $n \geq 1$, and $f \in L^1_{\text{loc}}(\Omega)$. We say that f is *weakly differentiable with respect to x_i* if there exists a function $g \in L^1_{\text{loc}}(\Omega)$ such that

$$\int_\Omega f\varphi_{x_i}\,dx = -\int_\Omega g\varphi\,dx \tag{2.2}$$

for any $\varphi \in C_c^\infty(\Omega)$. In this case, g is called the *weak derivative of f with respect to x_i*

Thus, the idea behind the definition of weak derivatives is to use the integration by parts formula as an abstract axiom. The advantage of defining weak derivatives in terms of integrals is that weak differentiability of a function is unaffected by the behavior of the function on a set of measure zero. This is because changing the value of a function on a set of measure zero does not change the value of the integral. Weak derivatives of higher-order can be defined in a similar way:

Definition 2.2. Let α be a multi-index of order $k \in \mathbb{N}$ and let $f \in L^1_{\text{loc}}(\Omega)$. We say that $g \in L^1_{\text{loc}}(\Omega)$ is the k^{th} *weak partial derivative of* f if

$$\int_\Omega f D^\alpha \varphi \, dx = (-1)^k \int_\Omega g\varphi \, dx$$

for all $\varphi \in C_c^\infty(\Omega)$. In this case, we write $D^\alpha f = g$.

Let us consider some examples of weak derivatives that illustrate the definition. If f is a function of a single variable, we use the familiar notation f' to denote the weak derivative of f.

Example 2.1. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbb{Q} \\ \sin(x) & \text{if} \quad x \notin \mathbb{Q}. \end{cases}$$

Clearly, f is discontinuous at almost every $x \in \mathbb{R}$. Hence it is not differentiable almost everywhere on $\mathbb{R}$. On the other hand, the function $g(x) := \cos(x)$ is a weak derivative of f on $\mathbb{R}$ since for any $\varphi \in C_c^\infty(\mathbb{R})$, we have

$$\int_{\mathbb{R}} f\varphi' \, dx = \int_{\mathbb{R}\setminus\mathbb{Q}} \sin(x)\varphi' \, dx = -\int_{\mathbb{R}\setminus\mathbb{Q}} \cos(x)\varphi \, dx = -\int_{\mathbb{R}} g\varphi \, dx.$$

Example 2.2. Consider the absolute value function $f(x) = |x|$ defined on $(-1, 1)$. Then f is not classically differentiable at $x = 0$. Let g be the *sign function* defined by

$$g(x) = \begin{cases} 1 & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad -1 < x < 0. \end{cases} \tag{2.3}$$

Then g is a weak derivative of f on the entire domain $(-1, 1)$ since for any $\varphi \in C_c^\infty(-1, 1)$, we have

$$\begin{aligned} \int_{-1}^{1} f\varphi' \, dx &= \int_{-1}^{0} f\varphi' \, dx + \int_{0}^{1} f\varphi' \, dx \\ &= \int_{-1}^{0} (-x)\varphi' \, dx + \int_{0}^{1} x\varphi' \, dx \\ &= \Big[-x\varphi \Big]_{-1}^{0} - \int_{-1}^{0} -\varphi \, dx + \Big[x\varphi \Big]_{0}^{1} - \int_{0}^{1} \varphi \, dx \\ &= -\varphi(-1) - \int_{-1}^{0} -\varphi \, dx + \varphi(1) - \int_{0}^{1} \varphi \, dx \\ &= -\int_{-1}^{1} g\varphi \, dx, \end{aligned}$$

where $\varphi(-1)$ and $\varphi(1)$ denote the values of φ at $x = -1$ and $x = 1$, respectively.

The previous two examples show that functions that are not differentiable in the ordinary sense can be weakly differentiable if the measure of the set of points at which the function is not differentiable is zero. Next example shows that a function with a jump discontinuity is not differentiable even in the weak sense. Intuitively, this is because disregarding the behavior of the function on a set of measure zero will not make the function differentiable in the ordinary sense on the entire domain of the function.

Example 2.3. Define a function $f : (-1, 1) \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad -1 < x \leq 0\,. \end{cases}$$

Then f does not have a weak derivative on $(-1,1)$. By contradiction, suppose it does. Then there exist $g \in L^1_{\text{loc}}(-1,1)$ such that

$$\int_{-1}^{1} f\varphi'\,dx = -\int_{-1}^{1} g\varphi\,dx$$

for any $\varphi \in C_c^\infty(-1,1)$. Since φ has compact support in $(-1,1)$, we have

$$\int_{-1}^{1} f\varphi'\,dx = \int_{0}^{1} \varphi'\,dx = -\varphi(0).$$

Hence, we must have

$$\int_{-1}^{1} g\varphi\,dx = \varphi(0) \tag{2.4}$$

for any $\varphi \in C_c^\infty(-1,1)$. We claim that this is not possible. To see why, choose a sequence of functions φ_k in $C_c^\infty(-1,1)$ such that $0 \le \varphi_k(x) \le 1$ for all $k \in \mathbb{N}$ and all x, $\lim\limits_{k\to\infty} \varphi_k(x) = 0$ for all x except at $x = 0$, and $\varphi_k(0) = 1$ for all $k \in \mathbb{N}$. Then by (2.4), we have

$$\int_{-1}^{1} g\varphi_k\,dx = \varphi_k(0) = 1 \tag{2.5}$$

for each $k \in \mathbb{N}$. Since $|g\varphi_k| \le |g| \in L^1_{\text{loc}}(-1,1)$ for all k, Lebesgue's Dominated Convergence Theorem (See Theorem A.3) implies $\lim\limits_{k\to\infty} \int_{-1}^{1} g\varphi_k\,dx = 0$. Thus, taking the limit $k \to \infty$ in (2.5), we get a contradiction.

We proceed to consider some basic properties of weak derivatives. First, we show that weak derivatives are unique up to a set of measure zero. To see this, let $f \in L^1_{\text{loc}}(\Omega)$ and suppose $g, \tilde{g} \in L^1_{\text{loc}}(\Omega)$ are two weak derivatives of f with respect to x_i. By definition,

$$\int_\Omega f\varphi_{x_i}\,dx = -\int_\Omega g\varphi\,dx$$

and

$$\int_\Omega f\varphi_{x_i}\,dx = -\int_\Omega \tilde{g}\varphi\,dx$$

for any $\varphi \in C_c^\infty(\Omega)$. Hence, we have

$$\int_\Omega g\varphi\, dx - \int_\Omega \tilde{g}\varphi\, dx = 0,$$

or

$$\int_\Omega (g - \tilde{g})\varphi\, dx = 0. \tag{2.6}$$

Now uniqueness follows by letting $f = g - \tilde{g}$ in the following lemma, taken from Wheeden and Zygmund [8], p. 463.

Lemma 2.1. *Let Ω be an open set in $\mathbb{R}^n$ and let $f \in L^1_{loc}(\Omega)$. If*

$$\int_\Omega f\varphi\, dx = 0$$

for all $\varphi \in C_c^\infty(\Omega)$, then $f = 0$ a.e. in Ω.

Suppose g is a weak derivative of f. Since weak derivatives are unique up to a set of measure zero, we call the equivalent class of g *the* weak derivative of f, where two functions are defined to be equivalent if they are equal almost everywhere. One can easily verify that if a function is differentiable in the ordinary sense, then its weak derivative corresponds to the ordinary derivative. For this reason, it is customary to use the notation f_{x_i} to denote the weak derivative of f even when f_{x_i} does not exist in the ordinary sense. Ordinary derivatives are also called *classical derivatives.*

Most properties of ordinary derivatives also hold for weak derivatives. Here we list some of them, which will be used in later sections. The proofs can be found in Evans [2] and Gilbarg and Trudinger [5].

Theorem 2.1. *Suppose $f, g \in L^1_{loc}(\Omega)$ are weakly differentiable with respect to x_i.*

(i) *(The product rule) If fg, $fg_{x_i} + f_{x_i}g \in L^1_{loc}(\Omega)$, then*

$$(fg)_{x_i} = fg_{x_i} + f_{x_i}g.$$

(ii) *(Linearity) For any* $\lambda, \mu \in \mathbb{R}$, $\lambda f + \mu g$ *is weakly differentiable with respect to* x_i *and*

$$(\lambda f + \mu g)_{x_i} = \lambda f_{x_i} + \mu g_{x_i}.$$

(iii) *(The chain rule) If* $h : \mathbb{R} \to \mathbb{R}$ *is continuously differentiable and its derivative is essentially bounded, then the composition* $h \circ f$ *is weakly differentiable with respect to* x_i *and*

$$(h \circ f)_{x_i} = (h' \circ f) f_{x_i}.$$

2.2 Sobolev Spaces

Definition 2.3. Let $1 \leq p \leq \infty$ and $k \in \mathbb{N}$. The *Sobolev space*, denoted by $W^{k,p}(\Omega)$, is defined to be the class of all locally integrable functions $f : \Omega \to \mathbb{R}$ such that for each multi-undex α satisfying $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(\Omega)$.

For each $1 \leq p \leq \infty$ and $k \in \mathbb{N}$, $W^{k,p}(\Omega)$ is a Banach space with respect to the norm given by

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\displaystyle\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p \, dx \right)^{\frac{1}{p}} & \text{if} \quad 1 \leq p < \infty \\ \displaystyle\sum_{|\alpha| \leq k} \operatorname*{ess\,sup}_{x \in \Omega} |D^\alpha u| & \text{if} \quad p = \infty. \end{cases}$$

The space $W_0^{k,p}(\Omega)$ is defined to be the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$. Thus, $u \in W_0^{k,p}(\Omega)$ if and only if $u \in C_c^\infty(\Omega)$ or there exists a sequence of functions in $C_c^\infty(\Omega)$ which converges to u with respect to the $W^{k,p}$ norm.

Example 2.4. Consider $f(x) = |x|$ on defined on $(-1, 1)$. We saw in Example 2.2 that the sign function g defined by (2.3) is the weak derivative of f. Since both f and its weak derivative g belong to $L^2(-1, 1)$, f is in $W^{1,2}((-1, 1))$.

In next chapter, we consider a certain type of a second-order PDE and see that the Sobolev space $W^{1,2}(\Omega)$ would be an appropriate function space in which to seek weak solutions. However, the problem with functions in $W^{1,2}(\Omega)$ is that they are defined only a.e. in Ω, and there is no obvious way to assign the values to u along $\partial\Omega$. For $x \in \partial\Omega$, a natural attempt would be to define $u(x) := a$ if $\lim_{y\to x} u(y)$ exists and is equal to a (here the limit $y \to x$ is to be taken through any path that y approaches to x from the interior of Ω), but this limit does not generally exist for non-smooth functions in $W^{1,2}(\Omega)$.

It turns out that there is a nice way out of this difficulty, which uses the following theorem taken from Evans [2] p. 268:

Theorem 2.2. *Let Ω be bounded with C^1-boundary. If $u \in W^{1,p}(\Omega)$ for some $1 \leq p < \infty$, then there exists a sequence of functions u_k in $C^\infty(\overline{\Omega})$ which converges to u with respect the $W^{1,p}$ norm.*

Theorem 2.2 is called *global approximation theorem* and helps us assign boundary values to functions in $W^{1,p}(\Omega)$. Let $u \in W^{1,p}(\Omega)$ and suppose $\{u_k\}$ is a sequence of functions in $C^\infty(\overline{\Omega})$ which converges to u with respect to the $W^{1,p}$ norm. If the boundary of Ω is sufficiently smooth (at least C^1), one can show that for each $k \in \mathbb{N}$ there is a unique continuous extension of u_k to $\overline{\Omega}$ (recall that originally functions in $C^\infty(\overline{\Omega})$ are only defined in Ω). For each $x \in \partial\Omega$, let us denote the value of the extension of u_k at x by $u_k|_{\partial\Omega}(x)$. Then the boundary values of $u \in W^{1,p}(\Omega)$, denoted by $Tu(x)$, is defined to be

$$Tu(x) := \lim_{k\to\infty} u_k|_{\partial\Omega}(x).$$

Next theorem, taken from Evans [2], p. 274, verifies the existence of Tu.

Theorem 2.3 (Trace Theorem)**.** *Let $1 \leq p \leq \infty$ and assume Ω is an open bounded* *subset of* $\mathbb{R}^n$ *with* C^1*-boundary. There exists a bounded linear operator*

$$T : W^{1,p}(\Omega) \to \; L^p(\partial\Omega)$$

such that

$$Tu = u|_{\partial\Omega} \quad \textit{if} \quad u \in C^1(\overline{\Omega})$$

and

$$\|Tu\|_{L^p(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$$

for each $u \in W^{1,p}(\Omega)$*. Here* C *is some constant which only depends on* p *and* n*, and* $u|_{\partial\Omega}$ *denotes the restriction of* u *to* $\partial\Omega$*.*

For each $u \in W^{1,p}(\Omega)$, Tu is called the *trace* of u on $\partial\Omega$. Next theorem gives us another interpretation of functions in $W_0^{1,p}(\Omega)$.

Theorem 2.4. *Assume* Ω *is an open bounded subset of* $\mathbb{R}^n$ *with* C^1*-boundary. Let* $1 \leq p \leq \infty$ *and suppose* $u \in W^{1,p}(\Omega)$*. Then* $u \in W_0^{1,p}(\Omega)$ *if and only if* $Tu = 0$ *on* $\partial\Omega$*.*

Chapter 3

Weak Solutions

As we mentioned in the introduction, there would be a need of extending the notion of solutions and including a larger class of functions as "weak solutions" when dealing with PDEs that exhibit non-smooth behaviors. In general, there is considerable freedom in how one defines weak solutions (i.e. the requirement for a function to be a weak solution is not given by the PDE itself), but typically we construct weak solutions in such a way that under some mild assumptions we obtain some degree of regularity for such solutions. In this chapter we formulate weak solutions to an eigenvalue problem with a mixed boundary condition.

Given a domain $\Omega \subset \mathbb{R}^n$, let D denote the Dirichlet part of the boundary and N denote the Neumann part of the boundary. We assume that D and N partition $\partial\Omega$. Define a space $C_D^\infty(\Omega)$ to be the class of all functions $u \in C^\infty(\Omega)$ such that $u = 0$ in a neighborhood of each x in D. For each $k \in \mathbb{N}$, the space $W_D^{k,p}(\Omega)$ is defined to be the closure of $C_D^\infty(\Omega)$ in $W^{k,p}(\Omega)$. Thus, $u \in W_D^{k,p}(\Omega)$ if and only if $u \in C_D^\infty(\Omega)$ or there exists a sequence of functions in $C_D^\infty(\Omega)$ which converges to u with respect to the $W^{k,p}$ norm.

It is not hard to show that the following version of Theorem 2.4 also holds for functions in $W_D^{1,p}(\Omega)$:

Theorem 3.1. *Assume Ω is an open bounded subset of $\mathbb{R}^n$ with C^1-boundary. Let $1 \leq p \leq \infty$ and suppose $u \in W^{1,p}(\Omega)$. Then $u \in W^{1,p}_D(\Omega)$ if and only if $Tu = 0$ on D.*

Let $u \in C^2(\Omega)$. We consider a second-order differential operator L given in the following divergence form

$$Lu = -\sum_{j=1}^{n}\sum_{i=1}^{n}\left(a_{ij}(x)u_{x_i}\right)_{x_j} \tag{3.1}$$

where $a_{ij}(x)$ are coefficient functions.

Definition 3.1. An operator L given by (3.1) is said to be *uniformly elliptic* if there exists a constant $\theta > 0$ such that

$$\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \tag{3.2}$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^n$.

The following is the simplest, but most important example of uniformly elliptic operators.

Example 3.1. Let L be given by (3.1) where

$$a_{ij} = \delta_{ij} := \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j\ . \end{cases}$$

Then $L = -\Delta$, which satisfies the ellipticity condition (3.2).

Let $\Omega \subset \mathbb{R}^n$ be a domain, where $n \geq 3$. We consider the eigenvalue problem:

$$\begin{aligned} Lu = \lambda u \quad &\text{in} \quad \Omega \\ u = 0 \quad &\text{on} \quad D \\ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}u_{x_i}\nu_j = f_N \quad &\text{on} \quad N. \end{aligned} \tag{3.3}$$

where L is given by (3.1) and the two boundary conditions are interpreted in the trace sense. We assume that L is uniformly elliptic and each coefficient function a_{ij} is measurable and essentially bounded. The nonhomogeneous Neumann data f_N is defined on the entire $\partial\Omega$ in such a way that $f_N = 0$ on D and $f_N \in L^{\frac{2(n-1)}{(n-2)}}(\partial\Omega)$ (note that $\frac{2(n-1)}{(n-2)} > 2$).

Definition 3.2. Suppose $u \in W_D^{1,2}(\Omega)$. We say that u is a *weak solution* or a *solution in the weak sense* to the eigenvalue problem (3.3) if

$$\sum_{j=1}^{n}\sum_{i=1}^{n}\int_{\Omega} a_{ij}u_{x_i}v_{x_j}\,dx = \lambda\int_{\Omega} uv\,dx + \int_{N} f_N v\,d\sigma, \tag{3.4}$$

for any $v \in W_D^{1,2}(\Omega)$. The formula (3.4) is called the *weak formulation* of the eigenvalue problem (3.3).

Note that the Hölder's inequality implies that each integral in (3.4) is finite whenever $u, v \in W_D^{1,2}(\Omega)$. The motivation of the above definition is as follows: suppose for the moment that a function u is in $C^2(\Omega)$ and solves (3.3). If we multiply the equation $Lu = \lambda u$ by an arbitrary smooth function $v \in C_D^{\infty}(\Omega)$, we get

$$(Lu)v = \lambda uv,$$

or

$$-\sum_{j=1}^{n}\sum_{i=1}^{n}(a_{ij}u_{x_i})_{x_j}v = \lambda uv.$$

Integrating both sides over Ω and using the integration by parts formula, we get

$$-\sum_{j=1}^{n}\sum_{i=1}^{n}\Big(-\int_{\Omega} a_{ij}u_{x_i}v_{x_j}\,dx + \int_{\partial\Omega} a_{ij}u_{x_i}v\nu_j\,d\sigma\Big) = \lambda\int_{\Omega} uv\,dx.$$

Since $v = 0$ near D, we have

$$-\sum_{j=1}^{n}\sum_{i=1}^{n}\Big(-\int_{\Omega} a_{ij}u_{x_i}v_{x_j}\,dx + \int_{N} a_{ij}u_{x_i}v\nu_j\,d\sigma\Big) = \lambda\int_{\Omega} uv\,dx.$$

Thus

$$\sum_{j=1}^{n}\sum_{i=1}^{n}\int_{\Omega} a_{ij}u_{x_i}v_{x_j}\,dx - \int_{N} f_N v\,d\sigma = \lambda\int_{\Omega} uv\,dx,$$

or

$$\sum_{j=1}^{n}\sum_{i=1}^{n}\int_{\Omega} a_{ij}u_{x_i}v_{x_j}\,dx = \lambda\int_{\Omega} uv\,dx + \int_{N} f_N v\,d\sigma. \tag{3.5}$$

Now, instead of assuming $u \in C^2(\Omega)$, let us see if there are any weaker assumptions under which (3.5) makes sense. First, to make sense of each integral in the summations, u must be at least weakly differentiable with respect to each x_i. Second, we must assume that each u_{x_i} is in $L^2(\Omega)$ to ensure that each integral in the summations is finite. Finally, in order to incorporate the Dirichlet boundary condition in (3.3), we require that $u = 0$ on D in the trace sense. By Theorem 3.1, we see that if u is in $W_D^{1,2}(\Omega)$, then u satisfies all of these conditions, which suggests that $W_D^{1,2}(\Omega)$ would be an appropriate function space in which to seek solutions to (3.3). Observe that our formulation of weak solutions does not require solutions to be differentiable in the ordinary sense. In fact, a solution only needs to have the first weak derivatives although this is a second order PDE. Hence, we see that this weak formulation allows us to include a much larger class of functions as weak solutions than the classical notion of solutions. The question is that – as we asked in the introduction – how nice are such weak solutions? At this point, all we know about regularity of weak solutions is that weak solutions and thier weak gradients belong to $L^2(\Omega)$. In next chapter, we will show that weak solutions to (3.3) in fact posses slightly higher regularity than this.

Chapter 4

A Reverse Hölder Inequality and Main Result

In this chapter, we assume Ω is a bounded, connected open subset of $\mathbb{R}^n$ with C^1-boundary, where $n \geq 3$. In addition, we assume that D is open relative to $\partial\Omega$ and satisfies the following condition: there exists $r_D > 0$ (which depends on D) such that for any $r \in (0, r_D)$, we get

$$\sigma(\Delta_r(x) \cap D) \geq C\sigma(\Delta_r(x)), \tag{4.1}$$

for all x in the closure of D with respect to $\partial\Omega$. Here, C is some constant such that $0 < C \leq 1$. This condition ensures that the surface measure of the Dirichlet part is comparable with that of the entire boundary.

4.1 Sobolev and Poincaré-Type Inequalities

Let $1 \leq p < n$. The *Sobolev conjugate of* p, denoted by p^*, is defined to be

$$p^* := \frac{np}{n-p}.$$

Note that

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n} \quad \text{and} \quad p^* > p.$$

The following is the classical Sobolev inequality:

Theorem 4.1 (Sobolev inequality). *If $u \in W_0^{1,p}(\Omega)$ for some $1 \leq p < n$, then*

$$\left(\int_\Omega |u|^q \, dx\right)^{\frac{1}{q}} \leq C \left(\int_\Omega |\nabla u|^p \, dx\right)^{\frac{1}{p}} \tag{4.2}$$

for each $q \in [1, p^*]$. Here the constant C only depends on p, q, n and Ω.

The Sobolev inequality shows that the integral of $|\nabla u|^p$ acts as an upper bound for the integral of $|u|^q$ where $q > p$. This means that whenever $u \in W_0^{1,p}(\Omega)$, u always gets more integrability than the gradient of u. In other words, Theorem 4.1 implies the following embedding:

$$W_0^{1,p}(\Omega) \subset L^q(\Omega) \quad \text{for each} \quad q \in [1, p^*].$$

It is a well-known result that if D is open in $\partial\Omega$ and satisfies (4.1), then we can replace $W_0^{1,p}(\Omega)$ in Theorem 4.1 with $W_D^{1,p}(\Omega)$ and the result still holds with a possibly different constant C.

We will also need the following Sobolev-type inequalities on Ω_r and Δ_r.

Theorem 4.2 (Sobolev-type inequality on Ω_r). *Fix $x^0 \in \Omega$ and $r > 0$. Let $1 \leq p \leq \infty$ and choose q so that $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. If $\overline{\Omega_r(x^0)} \cap D \neq \emptyset$, then for any $u \in W_D^{1,p}(\Omega)$, we get*

$$\int_{\Omega_r(x^0)} |u|^q \, dx \leq C\Big(\int_{\Omega_{2r}(x^0)} |\nabla u|^p \, dx\Big)^{\frac{q}{p}},$$

where the constant C only depends on n and p.

Theorem 4.3 (Sobolev-type inequality on Δ_r). *Fix $x^0 \in \Omega$ and $r > 0$. Let $1 \leq p < n$ and choose q so that $\frac{1}{q} = \frac{1}{p} - (1 - \frac{1}{p})(\frac{1}{n-1})$. If $\overline{\Omega_r(x^0)} \cap D \neq \emptyset$, then for any $u \in W_D^{1,p}(\Omega)$,* *we get*

$$\int_{\Delta_r(x^0)} |u|^q \, d\sigma \le C\Big(\int_{\Omega_{2r}(x^0)} |\nabla u|^p \, dx \Big)^{\frac{q}{p}},$$

where the constant C only depends on n and p.

Let $u : E \to \mathbb{R}$ be a measurable function. The *average of u over E*, denoted by $⨍_E u \, dx$, is defined by

$$⨍_E u \, dx := \frac{1}{|E|} \int_E u \, dx.$$

We also write $\overline{u}_E$ for $⨍_E u \, dx$.

Our assumptions on $\partial\Omega$ give us the following well-known Sobolev-Poincaré inequalities on Ω_r and Δ_r.

Theorem 4.4 (Sobolev-Poincaré Inequality on Ω_r)**.** *Let $1 \le p < n$ and choose q so that $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. If $u \in W^{1,p}(\Omega)$, then for any $x^0 \in \Omega$ and $r > 0$, we get*

$$\int_{\Omega_r(x^0)} |u - \bar{u}_{\Omega_r(x^0)}|^q \ dx \le C\Big(\int_{\Omega_r(x^0)} |\nabla u|^p \ dx \Big)^{\frac{q}{p}},$$

where C is a constant which only depends on n and p.

Theorem 4.5 (Sobolev-Poincaré Inequality on Δ_r)**.** *Let $1 \le p < n$ and choose q so that $\frac{1}{q} = \frac{1}{p} - (1 - \frac{1}{p})(\frac{1}{n-1})$. If $u \in W^{1,p}(\Omega)$, then for any $x^0 \in \Omega$ and $r > 0$, we get*

$$\int_{\Delta_r(x^0)} |u - \bar{u}_{\Omega_r(x^0)}|^q \, d\sigma \le C\Big(\int_{\Omega_r(x^0)} |\nabla u|^p \, dx \Big)^{\frac{q}{p}},$$

where C is a constant which only depends on n and p.

All the inequalities introduced in this section will be the essential tools in the proof of Theorem 4.8.

4.2 Main Result

Our main result is the following:

Theorem 4.6. *If $u \in W_D^{1,2}(\Omega)$ is a weak solution to the eigenvalue problem* (3.3)*, then there exists some $p > 2$ such that $u \in W_D^{1,p}(\Omega)$.*

To prove this theorem, we will show a reverse Hölder inequality for the gradient of eigenfunctions. First, recall the Hölder's inequality:

Theorem 4.7 (Hölder's inequality)**.** *Let $1 < p < \infty$ and let q denote the Hölder conjugate defined by*

$$q := \frac{p}{p-1}, \quad \textit{so that} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$, and

$$\int_\Omega |uv|\,dx \le \left(\int_\Omega |u|^p\,dx \right)^{1/p} \left(\int_\Omega |v|^q\,dx \right)^{1/q}.$$

In the special case $p = q = 2$, Hölder's inequality is known as Cauchy-Schwarz's inequality.

The Hölder's inequality says that the integral of $|u|^p$ acts as part of an upper bound for the integral of $|u|$ for $p > 1$. A reverse Hölder inequality refers to a type of inequalities where the integral of $|u|^p$ acts as part of an upper bound for the integral of $|u|^q$ for $q > p$. Our aim is to obtain a reverse Hölder inequality that gives an estimate for $\displaystyle\int_\Omega |\nabla u|^p\,dx$, where $u \in W_D^{1,2}(\Omega)$ is an eigenfunction and $p > 2$. To this end, we first prove the following theorem.

For any $f \in L^1_{\text{loc}}(\partial\Omega)$ and $r > 0$, define a local version of the Hardy-Littlewood maximal function $P_r f : \Omega \to \mathbb{R}$ by

$$P_r f(x) := \sup_{r>s>0} \fint_{\Delta_s(x)} |f(y)| d\sigma(y).$$

Theorem 4.8. *Suppose $u \in W_D^{1,2}(\Omega)$ is a weak solution to the eigenvalue problem (3.3). Then for any $x^0 \in \Omega$ and $r > 0$, we get*

$$\fint_{\Omega_r(x^0)} |\nabla u|^2\,dx \le C_1\left(\fint_{\Omega_{2r}(x^0)} |\nabla u|^{\frac{2n}{n+2}}\,dx\right)^{\frac{n+2}{n}} + C_2|\lambda|\fint_{\Omega_{2r}(x^0)} u^2\,dx + C_3\fint_{\Omega_{2r}(x^0)} P_r f_N^2\,dx + \frac{1}{2}\fint_{\Omega_{2r}(x^0)} |\nabla u|^2\,dx, \tag{4.3}$$

where C_1, C_2 and C_3 are positive constants only depending on $A := \sum_{j=1}^n\sum_{i=1}^n \operatorname{ess\,sup}_{x\in\Omega} |a_{ij}(x)|$, the dimension $n \ge 3$ and the ellipticity constant $\theta > 0$.

Proof. Fix $x^0 \in \Omega$ and $r > 0$. Define a a cutoff function $\varphi \in C_c^\infty(\mathbb{R}^n)$ by

$$\varphi(x) := \begin{cases} 1 & \text{if} \quad x \in B_r(x^0) \\ 0 & \text{if} \quad x \in \mathbb{R}^n \backslash B_{2r}(x^0) \end{cases}$$

and suppose φ satisfies $0 \le \varphi \le 1$ on $\mathbb{R}^n$ and $|\nabla\varphi| \le \frac{K_0}{r}$ on $B_{2r}(x^0)\backslash B_r(x^0)$ for some constant K_0 which only depends on n. Suppose $\rho \in \mathbb{R}$ is a constant such that $\varphi^2(u-\rho) \in W_D^{1,2}(\Omega)$. Below we will show that we can choose such ρ. By the weak formulation (3.4), we have

$$\sum_{j=1}^n\sum_{i=1}^n \int_\Omega a_{ij}u_{x_i}[\varphi^2(u-\rho)]_{x_j}\,dx = \lambda\int_\Omega u\varphi^2(u-\rho)\,dx + \int_N f_N\varphi^2(u-\rho)\,d\sigma.$$

By performing differentiation, we get

$$\sum_{j=1}^n\sum_{i=1}^n \int_\Omega a_{ij}u_{x_i}\varphi^2 u_{x_j}\,dx = -\sum_{j=1}^n\sum_{i=1}^n \int_\Omega 2a_{ij}u_{x_i}\varphi\varphi_{x_j}(u-\rho)\,dx + \lambda\int_\Omega u\varphi^2(u-\rho)\,dx + \int_N f_N\varphi^2(u-\rho)\,d\sigma.$$

Since L is uniformly elliptic, there exists a constant $\theta > 0$ such that

$$\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}u_{x_i}u_{x_j} \geq \theta|\nabla u|^2 \quad \text{a.e. on} \quad \Omega.$$

Thus,

$$\begin{aligned}
\int_\Omega \theta|\nabla u|^2\varphi^2\,dx &\leq \sum_{j=1}^{n}\sum_{i=1}^{n}\int_\Omega a_{ij}u_{x_i}u_{x_j}\varphi^2\,dx \\
&= -\sum_{j=1}^{n}\sum_{i=1}^{n}\int_\Omega 2a_{ij}u_{x_i}\varphi\varphi_{x_j}(u-\rho)\,dx + \lambda\int_\Omega u\varphi^2(u-\rho)\,dx \\
&\quad + \int_N f_N\varphi^2(u-\rho)\,d\sigma \\
&\leq \sum_{j=1}^{n}\sum_{i=1}^{n}\int_\Omega 2|a_{ij}||u_{x_i}|\varphi|\varphi_{x_j}||u-\rho|\,dx + |\lambda|\int_\Omega |u|\varphi^2|u-\rho|\,dx \\
&\quad + \int_N |f_N|\varphi^2|u-\rho|\,d\sigma \\
&\leq \int_\Omega 2A|\nabla u|\varphi\frac{K_0}{r}|u-\rho|\,dx + |\lambda|\int_\Omega |u|\varphi^2|u-\rho|\,dx + \int_N |f_N|\varphi^2|u-\rho|\,d\sigma,
\end{aligned}$$

where we used that $\sum_{j=1}^{n}\sum_{i=1}^{n}|a_{ij}| \leq A = \sum_{j=1}^{n}\sum_{i=1}^{n}\operatorname*{ess\,sup}_{x\in\Omega}|a_{ij}|$. Since $\varphi = 0$ outside the ball $B_{2r}(x^0)$, we have

$$\begin{aligned}
\int_{\Omega_{2r}(x^0)} \theta|\nabla u|^2\varphi^2\,dx \leq \int_{\Omega_{2r}(x^0)} 2A|\nabla u|\varphi\frac{K_0}{r}|u-\rho|\,dx + |\lambda|\int_{\Omega_{2r}(x^0)} |u|\varphi^2|u-\rho|\,dx \\
+ \int_{\Delta_{2r}(x^0)} |f_N|\varphi^2|u-\rho|\,d\sigma.
\end{aligned}$$

Now use Cauchy's inequality (See Proposition A.1) with $\varepsilon = \frac{\theta}{2}$, $a = \varphi|\nabla u|$, and $b = 2A\frac{K_0}{r}|u-\rho|$ to get

$$\begin{aligned}
\int_{\Omega_{2r}(x^0)} \theta|\nabla u|^2\varphi^2\,dx \leq \frac{\theta}{2}\int_{\Omega_{2r}(x^0)} \varphi^2|\nabla u|^2\,dx + \frac{2A^2K_0^2}{\theta r^2}\int_{\Omega_{2r}(x^0)} |u-\rho|^2\,dx \\
+ |\lambda|\int_{\Omega_{2r}(x^0)} |u|\varphi^2|u-\rho|\,dx + \int_{\Delta_{2r}(x^0)} |f_N|\varphi^2|u-\rho|\,d\sigma.
\end{aligned}$$

By subtracting the first term on the right from both sides, we have

$$\frac{\theta}{2}\int_{\Omega_{2r}(x^0)} |\nabla u|^2\varphi^2\,dx \le \frac{2A^2K_0^2}{\theta r^2}\int_{\Omega_{2r}(x^0)} |u-\rho|^2\,dx + |\lambda|\int_{\Omega_{2r}(x^0)} |u|\varphi^2|u-\rho|\,dx + \int_{\Delta_{2r}(x^0)} |f_N|\varphi^2|u-\rho|\,d\sigma.$$

Multiplying both sides by $\frac{2}{\theta}$ gives

$$\int_{\Omega_{2r}(x^0)} |\nabla u|^2\varphi^2\,dx \le \frac{4A^2K_0^2}{\theta^2 r^2}\int_{\Omega_{2r}(x^0)} |u-\rho|^2\,dx + \frac{2|\lambda|}{\theta}\int_{\Omega_{2r}(x^0)} |u|\varphi^2|u-\rho|\,dx + \frac{2}{\theta}\int_{\Delta_{2r}(x^0)} |f_N|\varphi^2|u-\rho|\,d\sigma. \tag{4.4}$$

Since

$$\int_{\Omega_r(x^0)} |\nabla u|^2\varphi^2\,dx \le \int_{\Omega_{2r}(x^0)} |\nabla u|^2\varphi^2\,dx$$

and $\varphi = 1$ on $\Omega_r(x^0)$, we have

$$\int_{\Omega_r(x^0)} |\nabla u|^2\,dx \le \int_{\Omega_{2r}(x^0)} |\nabla u|^2\varphi^2\,dx.$$

Also, since $0 \le \varphi^2 \le 1$ on $\mathbb{R}^n$, we have

$$\int_{\Omega_{2r}(x^0)} |u|\varphi^2|u-\rho|\,dx \le \int_{\Omega_{2r}(x^0)} |u||u-\rho|\,dx$$

and

$$\int_{\Delta_{2r}(x^0)} |f_N|\varphi^2|u-\rho|\,d\sigma \le \int_{\Delta_{2r}(x^0)} |f_N||u-\rho|\,d\sigma.$$

Thus, it follows from (4.4) that

$$\int_{\Omega_r(x^0)} |\nabla u|^2\,dx \le \frac{4A^2K_0^2}{\theta^2 r^2}\int_{\Omega_{2r}(x^0)} |u-\rho|^2\,dx + \frac{2|\lambda|}{\theta}\int_{\Omega_{2r}(x^0)} |u||u-\rho|\,dx + \frac{2}{\theta}\int_{\Delta_{2r}(x^0)} |f_N||u-\rho|\,d\sigma.$$

Using Cauchy's inequality with $a = |f_N|$, $b = |u - \rho|$, and an arbitrary $\varepsilon > 0$ (we will later choose such ε appropriately), we get

$$\begin{aligned}\int_{\Omega_r(x^0)} |\nabla u|^2\,dx \le \frac{4A^2K_0^2}{\theta^2 r^2}\int_{\Omega_{2r}(x^0)} |u-\rho|^2\,dx + \frac{2|\lambda|}{\theta}\int_{\Omega_{2r}(x^0)} |u||u-\rho|\,dx \\ + \frac{2\varepsilon}{\theta}\int_{\Delta_{2r}(x^0)} f_N^2\,d\sigma + \frac{1}{2\theta\varepsilon}\int_{\Delta_{2r}(x^0)} |u-\rho|^2\,d\sigma.\end{aligned} \tag{4.5}$$

Now we have two cases to consider.

Case 1: $B_{2r}(x^0) \cap D = \emptyset$.

In this case, we choose $\rho = \bar{u}_{\Omega_{2r}(x^0)}$. Then (4.5) becomes

$$\begin{aligned}\int_{\Omega_r(x^0)} |\nabla u|^2\,dx \le \frac{4A^2K_0^2}{\theta^2 r^2}\int_{\Omega_{2r}(x^0)} |u-\bar{u}_{\Omega_{2r}(x^0)}|^2\,dx + \frac{2|\lambda|}{\theta}\int_{\Omega_{2r}(x^0)} |u||u-\bar{u}_{\Omega_{2r}(x^0)}|\,dx \\ + \frac{2\varepsilon}{\theta}\int_{\Delta_{2r}(x^0)} f_N^2\,d\sigma + \frac{1}{2\theta\varepsilon}\int_{\Delta_{2r}(x^0)} |u-\bar{u}_{\Omega_{2r}(x^0)}|^2\,d\sigma.\end{aligned} \tag{4.6}$$

Note that the condition $\varphi = 0$ outside the ball $B_{2r}(x^0)$ and the property of weak derivatives (See Theorem 2.1 (ii)) imply that $\varphi^2(u - \bar{u}_{\Omega_{2r}(x^0)}) \in W_D^{1,2}(\Omega)$, so it is an appropriate test function. By Theorem A.1, we get

$$\int_{\Omega_{2r}(x^0)} |u||u-\bar{u}_{\Omega_{2r}(x^0)}|\,dx \le 2\int_{\Omega_{2r}(x^0)} u^2\,dx. \tag{4.7}$$

Choose the exponents in Theorem 4.4 to be $p = \frac{2n}{n+2}$ and $q = 2$. Then p and q satisfy $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$ and $1 \le p < 2$. Since $u \in W_D^{1,2}(\Omega) \subset W^{1,2}(\Omega) \subset W^{1,p}(\Omega)$, we get

$$\int_{\Omega_{2r}(x^0)} |u-\bar{u}_{\Omega_{2r}(x^0)}|^2\,dx \le K_1\left(\int_{\Omega_{2r}(x^0)} |\nabla u|^{\frac{2n}{n+2}}\,dx\right)^{\frac{n+2}{n}} \tag{4.8}$$

for some constant K_1 which only depends on n. Next choose the exponents in Theorem 4.5 to be $p = \frac{2n}{n+1}$ and $q = 2$. Then, p and q satisfy $\frac{1}{q} = \frac{1}{p} - (1-\frac{1}{p})(\frac{1}{n-1})$ and $1 \le p < 2$.

Since $u \in W^{1,p}(\Omega)$, we get

$$\int_{\Delta_{2r}(x^0)} |u - \bar{u}_{\Omega_{2r}(x^0)}|^2 \, d\sigma \leq K_2 \left(\int_{\Omega_{2r}(x^0)} |\nabla u|^{\frac{2n}{n+1}} \, dx \right)^{\frac{n+1}{n}} \tag{4.9}$$

for some constant K_2 which only depends on n. Note that by applying the Hölder's inequality with $p = \frac{n+1}{n}$ and $q = n+1$, we get

$$\begin{aligned} \int_{\Omega_{2r}(x^0)} |\nabla u|^{\frac{2n}{n+1}} \, dx &\leq \left(\int_{\Omega_{2r}(x^0)} |\nabla u|^2 \, dx \right)^{\frac{n}{n+1}} \left(\int_{\Omega_{2r}(x^0)} 1 \, dx \right)^{\frac{1}{n+1}} \\ &= \left(\int_{\Omega_{2r}(x^0)} |\nabla u|^2 \, dx \right)^{\frac{n}{n+1}} |\Omega_{2r}(x^0)|^{\frac{1}{n+1}}. \end{aligned}$$

Thus, it follows from (4.9) that

$$\int_{\Delta_{2r}(x^0)} |u - \bar{u}_{\Omega_{2r}(x^0)}|^2 \, d\sigma \leq K_2 \left(\int_{\Omega_{2r}(x^0)} |\nabla u|^2 \, dx \right) |\Omega_{2r}(x^0)|^{\frac{1}{n}}. \tag{4.10}$$

Using estimates (4.7), (4.8) and (4.10), it follows from (4.6) that

$$\begin{aligned} \int_{\Omega_r(x^0)} |\nabla u|^2 \, dx \leq{}& \frac{4A^2 K_0^2 K_1}{\theta^2 r^2} \left(\int_{\Omega_{2r}(x^0)} |\nabla u|^{\frac{2n}{n+2}} \, dx \right)^{\frac{n+2}{n}} + \frac{4|\lambda|}{\theta} \int_{\Omega_{2r}(x^0)} u^2 \, dx \\ &+ \frac{2\varepsilon}{\theta} \int_{\Delta_{2r}(x^0)} f_N^2 \, d\sigma + \frac{K_2 |\Omega_{2r}(x^0)|^{\frac{1}{n}}}{2\theta\varepsilon} \int_{\Omega_{2r}(x^0)} |\nabla u|^2 \, dx. \end{aligned}$$

Now dividing through by $|\Omega_r(x^0)|$ and noting that we can write $|\Omega_r(x^0)| = K_3 |\Omega_{2r}(x^0)|$ for some constant K_3 which only depends on n, we get

$$\begin{aligned} \fint_{\Omega_r(x^0)} |\nabla u|^2 \, dx \leq{}& \frac{4A^2 K_0^2 K_1 |\Omega_{2r}(x^0)|^{\frac{2}{n}}}{K_3 \theta^2 r^2} \left(\fint_{\Omega_{2r}(x^0)} |\nabla u|^{\frac{2n}{n+2}} \, dx \right)^{\frac{n+2}{n}} + \frac{4|\lambda|}{K_3 \theta} \fint_{\Omega_{2r}(x^0)} u^2 \, dx \\ &+ \frac{2\varepsilon}{K_3 \theta |\Omega_{2r}(x^0)|} \int_{\Delta_{2r}(x^0)} f_N^2 \, d\sigma + \frac{K_2 |\Omega_{2r}(x^0)|^{\frac{1}{n}}}{2K_3 \theta\varepsilon} \fint_{\Omega_{2r}(x^0)} |\nabla u|^2 \, dx. \end{aligned}$$

If $\Delta_{2r}(x^0) \neq \emptyset$, then we can write $|\Omega_{2r}(x^0)| = K_4 r \sigma(\Delta_{2r}(x^0))$ and $|\Omega_{2r}(x^0)| = K_5 r^n$

for some constants K_4 and K_5 which only depend on n. Thus,

$$\fint_{\Omega_r(x^0)} |\nabla u|^2\,dx \le \frac{4A^2K_0^2K_1K_5^{\frac{2}{n}}r^2}{K_3\theta^2r^2}\left(\fint_{\Omega_{2r}(x^0)} |\nabla u|^{\frac{2n}{n+2}}\,dx\right)^{\frac{n+2}{n}} + \frac{4|\lambda|}{K_3\theta}\fint_{\Omega_{2r}(x^0)} u^2\,dx + \frac{2\varepsilon}{K_3K_4\theta r}\fint_{\Delta_{2r}(x^0)} f_N^2\,d\sigma + \frac{K_2K_5^{\frac{1}{n}}r}{2K_3\theta\varepsilon}\fint_{\Omega_{2r}(x^0)} |\nabla u|^2\,dx,$$

or

$$\fint_{\Omega_r(x^0)} |\nabla u|^2\,dx \le \frac{4A^2K_0^2K_1K_5^{\frac{2}{n}}}{K_3\theta^2}\left(\fint_{\Omega_{2r}(x^0)} |\nabla u|^{\frac{2n}{n+2}}\,dx\right)^{\frac{n+2}{n}} + \frac{4|\lambda|}{K_3\theta}\fint_{\Omega_{2r}(x^0)} u^2\,dx + \frac{2\varepsilon}{K_3K_4\theta r}P_r f_N^2(x^0) + \frac{K_2K_5^{\frac{1}{n}}r}{2K_3\theta\varepsilon}\fint_{\Omega_{2r}(x^0)} |\nabla u|^2\,dx.$$

Choosing $\varepsilon = \frac{K_2K_5^{\frac{1}{n}}r}{K_3\theta}$ and using Proposition A.2, we obtain the desired result (4.3).

Case 2: $B_{2r}(x^0) \cap D \neq \emptyset$.

In this case, we choose $\rho = 0$. Then (4.5) becomes

$$\int_{\Omega_r(x^0)} |\nabla u|^2\,dx \le \frac{4A^2K_0^2}{\theta^2r^2}\int_{\Omega_{2r}(x^0)} u^2\,dx + \frac{2|\lambda|}{\theta}\int_{\Omega_{2r}(x^0)} u^2\,dx + \frac{2\varepsilon}{\theta}\int_{\Delta_{2r}(x^0)} f_N^2\,d\sigma + \frac{1}{2\theta\varepsilon}\int_{\Delta_{2r}(x^0)} u^2\,d\sigma. \tag{4.11}$$

Note that $\varphi^2 u \in W_D^{1,2}(\Omega)$, so it is an appropriate test function. Choose the exponents in Theorem 4.2 to be $p = \frac{2n}{n+2}$ and $q = 2$. Since $u \in W_D^{1,2} \subset W_D^{1,p}$ and $\Omega_{2r}(x^0) \cap D \neq \emptyset$, we get

$$\int_{\Omega_{2r}(x^0)} u^2\,dx \le K_6\left(\int_{\Omega_{4r}(x^0)} |\nabla u|^{\frac{2n}{n+2}}\,dx\right)^{\frac{n+2}{n}}, \tag{4.12}$$

where K_6 is some constant only depending on n. Choosing the exponents in Theorem

4.3 to be $p = \frac{2n}{n+1}$ and $q = 2$, we get

$$\int_{\Delta_{2r}(x^0)} u^2\,dx \le K_7\Big(\int_{\Omega_{4r}(x^0)} |\nabla u|^{\frac{2n}{n+1}}\,dx\Big)^{\frac{n+1}{n}} \le K_7\Big(\int_{\Omega_{4r}(x^0)} |\nabla u|^2\,dx\Big)|\Omega_{4r}(x^0)|^{\frac{1}{n}}, \tag{4.13}$$

where we applied the Hölder's inequality to obtain the second inequality. Using the estimates (4.12) and (4.13), it follows from (4.11) that

$$\begin{aligned}\int_{\Omega_r(x^0)} |\nabla u|^2\,dx \le \frac{4A^2K_0^2K_6}{\theta^2 r^2}\Big(\int_{\Omega_{4r}(x^0)} |\nabla u|^{\frac{2n}{n+2}}\,dx\Big)^{\frac{n+2}{n}} + \frac{2|\lambda|}{\theta}\int_{\Omega_{2r}(x^0)} u^2\,dx \\ + \frac{2\varepsilon}{\theta}\int_{\Delta_{2r}(x^0)} f_N^2\,d\sigma + \frac{K_7|\Omega_{4r}(x^0)|^{\frac{1}{n}}}{2\theta\varepsilon}\int_{\Omega_{4r}(x^0)} |\nabla u|^2\,dx.\end{aligned}$$

Using a similar argument as in Case 1 and a standard covering argument (see Theorem A.1), we obtain the desired result (4.3). □

Next theorem, taken from Giaquinta [4], p. 122 and slightly modified, shows that a function $g \in L^q(\Omega)$ indeed belongs to $L^p(\Omega)$ for some $p > q$ if the average of g^q over Ω_r do not exceed the average of g over Ω_{2r} for more than a fixed factor plus some appropriate terms.

Theorem 4.9. *Let $1 < q < \tilde{q}$ and suppose f and g are two nonnegative functions such that $f \in L^{\tilde{q}}(\Omega)$ and $g \in L^q(\Omega)$. If f and g satisfy*

$$\fint_{\Omega_r(x^0)} g^q\,dx \le K_1\left(\fint_{\Omega_{2r}(x^0)} g\,dx\right)^q + \fint_{\Omega_{2r}(x^0)} f^q\,dx + K_2\fint_{\Omega_{2r}(x^0)} g^q\,dx$$

for each $x^0 \in \Omega$ and each $r > 0$, where K_1 and K_2 are some constants such that $K_1 \ge 0$ and $0 \le K_2 < 1$, then there exists $\varepsilon > 0$ and a constant $C > 0$, both of which only depend on the dimension n, $q, \tilde{q}, K_1$ and K_2, such that $g \in L^p(\Omega)$ for each $p \in [q, q+\varepsilon)$ and we get the following reverse Hölder inequality

$$\left(\fint_{\Omega_r(x^0)} g^p\,dx\right)^{\frac{1}{p}} \le C\left[\left(\fint_{\Omega_{2r}(x^0)} g^q\,dx\right)^{\frac{1}{q}} + \left(\fint_{\Omega_{2r}(x^0)} f^p\,dx\right)^{\frac{1}{p}}\right], \tag{4.14}$$

for each $x^0 \in \Omega$ *and each* $r > 0$.

In order to apply the preceding theorem, we will need Lemma 4.2. To prove it, we borrow the following lemma from Ott and Brown [7].

Lemma 4.1. *Let* $p > 1$ *and choose* q *so that* $1 \le q \le \frac{pn}{n-1}$. *If* $f \in L^p_{loc}(\partial\Omega)$, *then for any* $x^0 \in \partial\Omega$ *and any* $r > 0$, *we have*

$$\left(\fint_{\Omega_r(x^0)} |P_r f|^q \, dx \right)^{\frac{1}{q}} \le C \left(\fint_{\Delta_{2r}(x^0)} |f|^p \, d\sigma \right)^{\frac{1}{p}},$$

where C *is a constant only depending on the dimension* n.

Lemma 4.2. *Let* $f_N \in L^{\frac{2(n-1)}{(n-2)}}(\partial\Omega)$ *be the nonhomogeneous Neumann data given in* (3.3). *Then* $(P_r f_N^2)^{\frac{1}{2}} \in L^{\frac{2n}{n-2}}(\Omega)$ *for any* $r > 0$.

Proof. Set $p = \frac{n-1}{n-2}$, $q = \frac{n}{n-2}$ and $f = f_N^2$ in Lemma 4.1. Then, $p > 1$ and $1 \le q \le \frac{pn}{n-1}$. Also, we have $f_N^2 \in L^{\frac{n-1}{n-2}}(\partial\Omega) \subset L^{\frac{n-1}{n-2}}_{\text{loc}}(\partial\Omega)$ since

$$\int_{\partial\Omega} (f_N^2)^{\frac{n-1}{n-2}} \, d\sigma = \int_{\partial\Omega} (f_N)^{\frac{2(n-1)}{n-2}} \, d\sigma < \infty$$

by the hypothesis. Thus, Lemma 4.1 implies that for any $x^0 \in \partial\Omega$ and any $r > 0$, we get

$$\left(\fint_{\Omega_r(x^0)} (P_r f_N^2)^{\frac{n}{n-2}} \, dx \right)^{\frac{n-2}{n}} \le C \left(\fint_{\Delta_{2r}(x^0)} (f_N^2)^{\frac{n-1}{n-2}} \, d\sigma \right)^{\frac{n-2}{n-1}},$$

or equivalently

$$\left(\fint_{\Omega_r(x^0)} \left[(P_r f_N^2)^{\frac{1}{2}} \right]^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} \le C \left(\fint_{\Delta_{2r}(x^0)} f_N^{\frac{2(n-1)}{n-2}} \, d\sigma \right)^{\frac{n-2}{n-1}}, \tag{4.15}$$

where C is a constant only depending on the dimension n. Since

$$\fint_{\Delta_{2r}(x^0)} f_N^{\frac{2(n-1)}{n-2}} \, d\sigma < \infty$$

by the hypothesis, it follows from (4.15) that

$$\left(\fint_{\Omega_r(x^0)} \left[(P_r f_N^2)^{\frac{1}{2}}\right]^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}} < \infty,$$

which gives the desired result. □

Now we are ready to prove our main result.

Proof of Theorem 4.6. Suppose $u \in W_D^{1,2}(\Omega)$ is a weak solution to (3.3). Then, by Theorem 4.8 we have

$$\begin{aligned}\fint_{\Omega_r(x^0)} |\nabla u|^2 \, dx \leq C_1 \left(\fint_{\Omega_{2r}(x^0)} |\nabla u|^{\frac{2n}{n+2}} dx\right)^{\frac{n+2}{n}} + C_2|\lambda| \fint_{\Omega_{2r}(x^0)} u^2 \, dx \\ + C_3 \fint_{\Omega_{2r}(x^0)} P_r f_N^2 \, dx + \frac{1}{2} \fint_{\Omega_{2r}(x^0)} |\nabla u|^2 \, dx\end{aligned} \tag{4.16}$$

for any $x^0 \in \Omega$ and any $r > 0$. Here C_1, C_2 and C_3 are some positive constants which only depend on $A := \sum_{j=1}^{n} \sum_{i=1}^{n} \operatorname*{ess\,sup}_{x\in\Omega} |a_{ij}(x)|$, n and the ellipticity constant $\theta > 0$. Observe that we can write

$$\begin{aligned}C_2|\lambda| \fint_{\Omega_{2r}(x^0)} u^2 \, dx + C_3 \fint_{\Omega_{2r}(x^0)} P_r f_N^2 \, dx &= \fint_{\Omega_{2r}(x^0)} (C_2|\lambda| u^2 + C_3 P_r f_N^2) \, dx \\ &= \fint_{\Omega_{2r}(x^0)} \left[\left(C_2^{\frac{1}{2}} |\lambda|^{\frac{1}{2}} |u|\right)^2 + \left(C_3^{\frac{1}{2}} (P_r f_N^2)^{\frac{1}{2}}\right)^2\right] dx \\ &\leq \fint_{\Omega_{2r}(x^0)} \left(C_2^{\frac{1}{2}} |\lambda|^{\frac{1}{2}} |u| + C_3^{\frac{1}{2}} (P_r f_N^2)^{\frac{1}{2}}\right)^2 dx \\ &= \fint_{\Omega_{2r}(x^0)} \left(C_2^{\frac{1}{2}} |\lambda|^{\frac{1}{2}} |u| + C_3^{\frac{1}{2}} (P_r f_N^2)^{\frac{1}{2}}\right)^{\frac{2n}{n+2} \cdot \frac{n+2}{n}} dx.\end{aligned}$$

Thus, we can rewrite the inequality (4.16) as

$$\fint_{\Omega_r(x^0)} |\nabla u|^2\,dx \le C_1\left(\fint_{\Omega_{2r}(x^0)} |\nabla u|^{\frac{2n}{n+2}}\,dx\right)^{\frac{n+2}{n}} + \fint_{\Omega_{2r}(x^0)} \left(C_2^{\frac12}|\lambda|^{\frac12}|u| + C_3^{\frac12}(P_r f_N^2)^{\frac12}\right)^{\frac{2n}{n+2}\cdot\frac{n+2}{n}} dx + \frac12\fint_{\Omega_{2r(x^0)}} |\nabla u|^2\,dx. \tag{4.17}$$

To apply Theorem 4.9, set $q = \frac{n+2}{n}$, $\tilde q = \frac{n+2}{n-2}$, $f = \left(C_2^{\frac12}|\lambda|^{\frac12}|u| + C_3^{\frac12}(P_r f_N^2)^{\frac12}\right)^{\frac{2n}{n+2}}$, and $g = |\nabla u|^{\frac{2n}{n+2}}$. Then $1 < q < \tilde q$ and we have

$$\begin{aligned}\int_\Omega |f|^{\tilde q}\,dx &= \int_\Omega \left(C_2^{\frac12}|\lambda|^{\frac12}|u| + C_3^{\frac12}(P_r f_N^2)^{\frac12}\right)^{\frac{2n}{n+2}\cdot\frac{n+2}{n-2}} dx\\ &= \int_\Omega \left(C_2^{\frac12}|\lambda|^{\frac12}|u| + C_3^{\frac12}(P_r f_N^2)^{\frac12}\right)^{\frac{2n}{n-2}} dx.\end{aligned}$$

Note that $\frac{2n}{n-2}$ is the Sobolev conjugate of 2. Since $u \in W_D^{1,2}(\Omega)$, Theorem 4.1 implies that $|u| \in L^{\frac{2n}{n-2}}(\Omega)$. Moreover, from Lemma 4.2 we know $(P_r f_N^2)^{\frac12} \in L^{\frac{2n}{n-2}}(\Omega)$. Since Lebesgue spaces are vector spaces, it follows that

$$\int_\Omega |f|^{\tilde q}\,dx = \int_\Omega \left(C_2^{\frac12}|\lambda|^{\frac12}|u| + C_3^{\frac12}(P_r f_N^2)^{\frac12}\right)^{\frac{2n}{n-2}} dx < \infty$$

Hence $f \in L^{\tilde q}(\Omega)$. Also, since

$$\int_\Omega |g|^q\,dx = \int_\Omega |\nabla u|^{\frac{2n}{n+2}\cdot\frac{n+2}{n}}\,dx = \int_\Omega |\nabla u|^2\,dx < \infty,$$

we have $g \in L^q(\Omega)$. Now with these choices for q, $\tilde q$, f, and g, the inequality (4.17) becomes

$$\fint_{\Omega_r(x^0)} g^q\,dx \le C_1\left(\fint_{\Omega_{2r}(x^0)} g\,dx\right)^q + \fint_{\Omega_{2r}(x^0)} f^q\,dx + \frac12\fint_{\Omega_{2r}(x^0)} g^q\,dx. \tag{4.18}$$

Since (4.18) holds for any $x^0 \in \Omega$ and any $r > 0$, Theorem 4.9 implies that there exists

$\varepsilon > 0$ such that $g = |\nabla u|^{\frac{2n}{n+2}} \in L^p(\Omega)$ for each $p \in [\frac{n+2}{n}, \frac{n+2}{n} + \varepsilon)$. Setting $\tilde{p} = \frac{2np}{n+2}$ and $\tilde{\varepsilon} = \frac{2n\varepsilon}{n+2}$, we see that this is equivalent to $|\nabla u| \in L^{\tilde{p}}(\Omega)$ for each $\tilde{p} \in [2, 2 + \tilde{\varepsilon})$. This completes the proof.

$\square$

Appendix

Theorem A.1. *Let $\Omega \subset \mathbb{R}^n$ be open and suppose $x^0 \in \Omega$. For two nonnegative functions u and v defined on Ω, if*

$$\int_{\Omega_r(x^0)} u\,dx \leq \int_{\Omega_{4r}(x^0)} v\,dx$$

for any $r > 0$, then there exists a constant C only depending on n such that

$$\int_{\Omega_r(x^0)} u\,dx \leq C \int_{\Omega_{2r}(x^0)} v\,dx$$

for any $r > 0$.

Proof. For any $r > 0$, let $s = \frac{r}{4}$. Let $\mathcal{A} := \{\Omega_s(x) : x \in \Omega_r(x^0)\}$. Then $\mathcal{A}$ is an open cover of $\overline{\Omega_r(x^0)}$. Since $\overline{\Omega_r(x^0)}$ is compact, there exists a finite subcover of $\mathcal{A}$, which we denote by $\mathcal{A}' := \{\Omega_s(x^k) : x^k \in \Omega_r(x^0) \text{ for } k = 1, 2, \ldots, m\}$. Let $B := \bigcup_{k=1}^{m} \Omega_s(x^k)$. Then $\Omega_r(x^0) \subset B$. By hypothesis, we have

$$\int_{\Omega_s(x^k)} u\,dx \leq \int_{\Omega_{4s}(x^k)} u\,dx = \int_{\Omega_r(x^k)} u\,dx.$$

for any x^k. Also, note that for any k we have

$$\Omega_r(x^k) \subset \Omega_{2r}(x^0). \tag{A.1}$$

Since

$$\int_{\Omega_r(x^0)} u\,dx \leq \int_B u\,dx \leq \sum_{k=1}^{m} \int_{\Omega_s(x^k)} u\,dx \leq \sum_{k=1}^{m} \int_{\Omega_r(x^k)} v\,dx,$$

it follows from (A.1) that

$$\int_{\Omega_r(x^0)} u\,dx \leq \sum_{k=1}^{m} \int_{\Omega_{2r}(x^0)} v\,dx = m \int_{\Omega_{2r}(x^0)} v\,dx.$$

Note that m only depends on n. □

Theorem A.2 (Minkowski's inequality)**.** *Let* $1 \leq p \leq \infty$*. If* $u, v \in L^p(\Omega)$*, then* $u + v \in L^p(\Omega)$ *and*

$$\|u+v\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}.$$

Proof. See Wheeden and Zygmund [8] p. 188. □

Proposition A.1 (Cauchy's inequality)**.** *For any* $a, b \in \mathbb{R}$ *and any* $\varepsilon > 0$*, we have*

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}.$$

Proof. We have

$$0 \leq (a-b)^2 = a^2 - 2ab + b^2,$$

so

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}. \tag{A.2}$$

Let $\alpha = a\sqrt{2\varepsilon}$ and $\beta = \left(\frac{b}{\sqrt{2\varepsilon}}\right)$. Then by (A.2), we get

$$ab = \alpha\beta \leq \frac{\alpha^2}{2} + \frac{\beta^2}{2} = a^2\varepsilon + \frac{b^2}{4\varepsilon}.$$

□

Lemma A.1. *Let* $\Omega \subset \mathbb{R}^n$ *be open and suppose* $u \in L^2(\Omega)$*. Then, for any* $r > 0$ *and* $x^0 \in \Omega$*, we have*

$$\int_{\Omega_r(x^0)} |u||u - \bar{u}_{\Omega_r(x^0)}|\, dx \leq 2 \int_{\Omega_r(x^0)} u^2\, dx.$$

Proof. We have

$$\begin{aligned} |u||u - \bar{u}_{\Omega_r(x^0)}| &\leq |u^2 - \bar{u}_{\Omega_r(x^0)}u| \leq |u^2| + |\bar{u}_{\Omega_r(x^0)}u| \\ &\leq u^2 + \frac{|u|}{|\Omega_r(x^0)|} \int_{\Omega_r(x^0)} |u(z)|\, dz. \end{aligned}$$

Hence,

$$\begin{aligned}
\int_{\Omega_r(x^0)} |u||u - \bar{u}_{\Omega_r(x^0)}|\,dx &\le \int_{\Omega_r(x^0)} \left(u^2 + \frac{|u|}{|\Omega_r(x^0)|} \int_{\Omega_r(x^0)} |u(z)|\,dz \right) dx \\
&= \int_{\Omega_r(x^0)} u^2\,dx + \int_{\Omega_r(x^0)} \left(\frac{|u|}{|\Omega_r(x^0)|} \int_{\Omega_r(x^0)} u(z)dz \right) dx \\
&= \int_{\Omega_r(x^0)} u^2\,dx + \frac{1}{|\Omega_r(x^0)|} \int_{\Omega_r(x^0)} |u|\,dz \int_{\Omega_r(x^0)} |u|\,dx \\
&\le \int_{\Omega_r(x^0)} u^2\,dx + \frac{1}{|\Omega_r(x^0)|} \left(\int_{\Omega_r(x^0)} |u|\,dx \right)^2.
\end{aligned}$$

By Cauchy-Schwarz inequality, we have

$$\left(\int_{\Omega_r(x^0)} |u|\,dx \right)^2 \le \int_{\Omega_r(x^0)} u^2\,dx \int_{\Omega_r(x^0)} 1\,dx.$$

Thus,

$$\begin{aligned}
\int_{\Omega_r(x^0)} |u||u - u_{\Omega_r(x^0)}|\,dx &\le \int_{\Omega_r(x^0)} u^2\,dx + \frac{1}{|\Omega_r(x^0)|} \int_{\Omega_r(x^0)} u^2\,dx \int_{\Omega_r(x^0)} 1\,dx \\
&= \int_{\Omega_r(x^0)} u^2\,dx + \int_{\Omega_r(x^0)} u^2\,dx \\
&= 2 \int_{\Omega_r(x^0)} u^2\,dx.
\end{aligned}$$

□

Lemma A.2. *If $f \in L^1_{loc}(\partial\Omega)$ is nonnegative, then there exists a constant C only depending on n such that*

$$P_r f(x^0) \le C \fint_{\Omega_r(x^0)} P_r f(x)\,dx$$

for any $x^0 \in \Omega$ and $r > 0$.

Proof. Let $x^0 \in \Omega$ and $s > 0$. Then for any $x \in \Omega_s(x^0)$, we have $\Delta_s(x^0) \subset \Delta_{2s}(x)$. Since f is nonnegative, it follows that

$$\int_{\Delta_s(x^0)} f\,dy \le \int_{\Delta_{2s}(x)} f\,dy.$$

Dividing both sides by $\sigma(\Delta_s(x^0))$, we get

$$\frac{1}{\sigma(\Delta_s(x^0))}\int_{\Delta_s(x^0)} f\,dy \le \frac{1}{\sigma(\Delta_s(x^0))}\int_{\Delta_{2r}(x^0)} dy,$$

or

$$⨍_{\Delta_s(x^0)} f\,dy \le \frac{1}{C}⨍_{\Delta_{2s}(x)} f\,dy, \tag{A.3}$$

where we used the fact that $\sigma(\Delta_s(x^0)) = C\sigma(\Delta_{2s}(x))$ for some constant C only depending on n. Fix $r > 0$. Taking the supremum over $s \in (0, r)$ on both sides of (A.3), we get

$$P_r f(x^0) \le \tilde{C} P_r f(x),$$

where $\tilde{C} = \frac{1}{C}$. Since this holds for all $x \in \Omega_r(x^0)$, we get

$$\int_{\Omega_r(x^0)} P_r f(x^0)\,dx \le \tilde{C}\int_{\Omega_r(x^0)} P_r f(x)\,dx.$$

The results follows if we observe that

$$P_r f(x^0) = P_r f(x^0)\frac{1}{\sigma(\Omega_r(x^0))}\int_{\Omega_r(x^0)} 1\,dx = ⨍_{\Omega_r(x^0)} P_r f(x^0)\,dx.$$

□

Theorem A.3 (Lebesgue's Dominated Convergence Theorem). *Let $E \subset \mathbb{R}^n$ be a measurable set and let $\{f_k\}$ be a sequence of measurable functions on E such $\lim_{k\to\infty} f_k(x) = f(x)$ a.e. on E. If there exists $g \in L(E)$ such that $|f_k(x)| \le |g(x)|$ a.e. on E for all $k \in \mathbb{N}$, then $\lim_{x\to\infty}\int_E f_k(x) = \int_E f(x)$.*

Proof. See Wheeden and Zygmund [8] p. 96. □

References

[1] A. Bensoussan and J. Frehse. *Regularity Results for Nonlinear Elliptic Systems and Applications*. Springer-Verlag Berlin Heidelberg, 2010.

[2] L. C. Evans. *Partial Differential Equations*. Corrected reprint of the second (2010) edition. Amer. Math. Soc., Providence, RI, 2015.

[3] F.W. Gehring. *The L^p-integrability of the Partial Derivatives of a Quasiconformal Mapping*. Acta Math., 130:265-277, 1973.

[4] M. Giaquinta. *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*. Princeton Univ. Press, 1983.

[5] D. Gilbarg and N. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Reprint of the 1998 edition. Springer-Verlag Berlin Heidelberg, 2001.

[6] D. Graffi. *Materials with Memory*. Reprint of the first edition. Springer-Verlag Berlin Heidelberg, 2011.

[7] K. A. Ott and R. M. Brown. *The mixed problem for the Laplacian in Lipschitz domains*. Potential Anal (2013) 38: 1333.

[8] R. L. Wheeden and A. Zygmund. *Measure and Integral: An Introduction to Real Analysis*. Second edition. CRS Press, Boca Raton, FL, 2015.